Short-Wave Limit of Hydrodynamics: A Soluble Example

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The Chapman-Enskog series for shear stress is summed up in a closed form for a simple model of Grad moment equations. The resulting linear hydrodynamics is demonstrated to be stable for all wavelengths, and the exact asymptotic of the acoustic spectrum in the short-wave domain is obtained. [S0031-9007(96)00642-4]

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A derivation of hydrodynamics from the Boltzmann kinetic equation is the classical problem of physical kinetics. The Chapman-Enskog (CE) method [1] gives, in principle, a possibility to compute a solution as a formal series in powers of Knudsen number ϵ (where ϵ is a ratio between the mean free path of a particle and the scale of variations of hydrodynamic quantities, density, mean flux, and temperature). The CE solution leads to a formal expansion of stress tensor and of heat flux vector in balance equations for density, momentum, and energy. Retaining the first order term (ϵ) in the latter expansions, we come to the Navier-Stokes equations, while further corrections are known as the Burnett (ϵ^2) and the super-Burnett (ϵ^3) corrections [1].

However, as demonstrated by Bobylev [2], even in the simplest regime (one-dimensional linear deviation from global equilibria), the Burnett and super-Burnett hydrodynamics violate the basic physics behind the Boltzmann equation. Namely, sufficiently short acoustic waves are increasing with time instead of decaying. This contradicts the H theorem, since all near-equilibrium perturbations must decay. It should also be noted that the instability of equilibria just mentioned is not a feature of the Navier-Stokes approximation where waves of arbitrary length are decaying, though this approximation is formally not valid in a short-wave domain. A possible root of this violation is poor convergency properties of CE series, and this, in particular, creates serious difficulties for an extension of hydrodynamics, as derived from a microscopic description, into a highly nonequilibrium domain. The latter problem remains one of the central open problems of the Boltzmann equation theory, in particular, and of the physical kinetics, in general.

In this Letter we consider the CE procedure for a simple model of nonhydrodynamic description (one-dimensional linearized 10-moment Grad equations [3]). The CE series, which is due to a *nonlinear* procedure even here and which also suffers the Bobylev instability in low-order approximations, is summed up in a closed form. This result leads to a quantitative discussion of the CE solution in a short-wave domain in frames of the model, and to a preliminary discussion of what can be expected in more realistic models. Exact results on the CE method for other Grad moment systems will be reported elsewhere.

Throughout the Letter, p and u are dimensionless deviations of pressure and of mean flux from their equilibrium values, respectively (see Ref. [4] for relations of these variables to dimensional quantities). The point of departure is the set of linearized Grad equations [4] for p, u, and σ, where σ is a dimensionless xx component of stress tensor,

$$\begin{aligned} \partial_t p &= -\frac{5}{3}\,\partial_x u\,, \\ \partial_t u &= -\partial_x p - \partial_x \sigma\,, \\ \partial_t \sigma &= -\frac{4}{3}\,\partial_x u - \frac{1}{\epsilon}\,\sigma\,. \end{aligned} \tag{1}$$

Equation (1) provides the simplest model of a coupling of the hydrodynamic variables, u and p, to the nonhydrodynamic variable σ, and corresponds to a heat nonconductive case. Of course, Eq. (1) is almost trivial. However, our goal here is not to investigate its properties but to *shorten* the description, and to get a closed set of equations with respect to variables p and u only. That is, we have to express the function σ in terms of spatial derivatives of p and of u. The CE method as applied to Eq. (1) results in the following [5]:

$$\sigma_{\mathrm{CE}} = \sum_{n=0}^{\infty} \epsilon^{n+1}\sigma^{(n)}. \tag{2}$$

The coefficients $\sigma^{(n)}$ are due to the following recurrence procedure [4]:

$$\sigma^{(n)} = -\sum_{m=0}^{n-1} \partial_t^{(m)} \sigma^{(n-1-m)}, \tag{3}$$

where the CE operators $\partial_t^{(m)}$ act on p, on u, and on their spatial derivatives as follows:

$$\begin{aligned} \partial_t^{(m)}\partial_x^l u &= \begin{cases} -\partial_x^{l+1} p, & m = 0\,, \\ -\partial_x^{l+1}\sigma^{(m-1)}, & m \geq 1\,, \end{cases} \\ \partial_t^{(m)}\partial_x^l p &= \begin{cases} -\frac{5}{3}\partial_x^{l+1} u, & m = 0\,, \\ 0, & m \geq 1\,. \end{cases} \end{aligned} \tag{4}$$

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Here $l \geq 0$ is an arbitrary integer, and $\partial_x^0 = 1$. Finally, $\sigma^{(0)} = -\frac{4}{3}\partial_x u$, which leads to the Navier-Stokes approximation.

Because of a somewhat involved structure of the recurrence procedure (3) and (4), the CE method is a nonlinear operation even in the simplest model (1). Moreover, as was shown in [4], the Bobylev instability is present: the Navier-Stokes and Burnett approximations are stable, while the super-Burnett approximation is unstable for sufficiently short waves.

Our goal now is to *sum up* the series (2) in a closed form. First, we will make some preparations. As was demonstrated in [4], the functions $\sigma^{(n)}$ in Eqs. (2) and (3) have the following structure for arbitrary $n \geq 0$:

$$\sigma^{(2n)} = a_n \partial_x^{2n+1} u, \qquad \sigma^{(2n+1)} = b_n \partial_x^{2(n+1)} p\,, \tag{5}$$

where numbers a_n and b_n are due to the recurrent procedure (3) and (4). Further, it is convenient to make the Fourier transform. Using a new space-time scale, $x' = \epsilon^{-1}x$, and $t' = \epsilon^{-1}t$, and next representing $u = \hat{u}\varphi(x', t')$, $p = \hat{p}\varphi(x', t')$, where $\varphi(x', t') = \exp(\omega t' + ikx')$, and k is a real-valued wave vector, we obtain $\sigma_{\mathrm{CE}} = \hat{\sigma}_{\mathrm{CE}}\varphi(x', t')$, where

$$\hat{\sigma}_{\mathrm{CE}}(k) = ikA(k^2)\hat{u} - k^2B(k^2)\hat{p} \tag{6}$$

and

$$A(k^2) = \sum_{n=0}^{\infty} a_n(-k^2)^n, \qquad B(k^2) = \sum_{n=0}^{\infty} b_n(-k^2)^n. \tag{7}$$

Thus, the question of summation of the CE series (2) amounts to finding the two functions, $A(k^2)$ and $B(k^2)$ (7). Knowing A and B, we derive a dispersion relation for acoustic waves, $\omega(k)$, upon a substitution of the function $\sigma_{\mathrm{CE}} = \hat{\sigma}_{\mathrm{CE}}\varphi$ into the second of Eqs. (1), and from a condition of a nontrivial solubility of a set of two linear equations with respect to $\hat{u}$ and $\hat{p}$. The result of these standard manipulations reads

$$\omega_{\pm} = \frac{k^2A}{2} \pm \frac{|k|}{2}\sqrt{k^2A^2 - \frac{20}{3}(1 - k^2B)}\,. \tag{8}$$

Now we will concentrate on a problem of a computation of the functions A and B (7) in a closed form. Substituting Eq. (5) into Eqs. (3) and (4), after some computations, we arrive at the following recurrence equations for the coefficients a_n and b_n in the power series (7):

$$\begin{aligned} a_{n+1} &= \frac{5}{3}b_n + \sum_{m=0}^{n} a_{n-m}a_m\,, \\ b_{n+1} &= a_{n+1} + \sum_{m=0}^{n} a_{n-m}b_m\,. \end{aligned} \tag{9}$$

The initial condition for this set of equations is $a_0 = -\frac{4}{3}$ and $b_0 = -\frac{4}{3}$.

At this point, it is worthwhile to notice that usual routes of dealing with the recurrence system (9) would be either to truncate it at a certain n, or to calculate all the coefficients explicitly, and next to substitute the result into the power series (7). Both these routes are not successful here. Indeed, retaining the coefficients a_0, b_0, and a_1 gives the super-Burnett approximation which has the short-wave instability for $k^2 > 3$ [4], and there is no guarantee that the same will not occur in a higher-order truncation. On the other hand, a term-by-term computation of the whole set of coefficients a_n and b_n is a quite nontrivial task due to a nonlinearity in Eq. (9).

Fortunately, another route is possible. Multiplying both the equations in (9) with $(-k^2)^{n+1}$, and performing a summation in n from zero to infinity, we get

$$A - a_0 = -k^2\left\{\frac{5}{3}B + \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m}a_m(-k^2)^m\right\},$$
$$B - b_0 = A - a_0 - k^2\sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m}b_m(-k^2)^m.$$

Now we notice that

$$\lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m}a_m(-k^2)^m = A^2,$$
$$\lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m}b_m(-k^2)^m = AB\,.$$

Accounting $a_0 = b_0 = -\frac{4}{3}$, we come to a pair of coupled quadratic equations for functions A and B,

$$A = -\frac{4}{3} - k^2(\tfrac{5}{3}B + A^2), \qquad B = A(1 - k^2B)\,. \tag{10}$$

The result (10) concludes essentially the question of computation of functions A and B (7). Still, further simplifications are possible. In particular, it is convenient to reduce a consideration to a single function. Resolving the system (10) with respect to B, and introducing a new function, $C(k^2) = k^2B(k^2)$, we get an equivalent cubic equation,

$$-\frac{5}{3}(C - 1)^2\left(C + \frac{4}{5}\right) = \frac{C}{k^2}\,. \tag{11}$$

Since functions A and B (7) are real valued, we are interested only in real-valued roots of Eq. (11). Elementary analysis of this equation gives the following result: *the real-valued root $C(k^2)$ is unique and negative for all finite values of parameter k^2.* Moreover, the function $C(k^2)$ is a monotonic function of k^2. Limiting values are

$$\lim_{|k|\to 0} C(k^2) = 0, \qquad \lim_{|k|\to\infty} C(k^2) = -\frac{4}{5}. \tag{12}$$

The function $C(k^2)$ is plotted in Fig. 1.

Under the circumstances just mentioned, a function under the root in Eq. (8) is negative for all k, including the limits, and we come to the following dispersion law:

$$\omega_\pm = \frac{C}{2(1-C)} \pm i\frac{|k|}{2}\sqrt{\frac{5C^2 - 16C + 20}{3}}, \tag{13}$$

where $C = C(k^2)$ is the real-valued root of Eq. (11), and $i = \sqrt{-1}$. Since $C(k^2)$ is a negative function for all $|k| > 0$, the damping rate, $\Re\omega_\pm$, is negative for all $|k| > 0$, and the exact acoustic spectrum of the CE procedure *is stable for arbitrary wavelengths.* In the short-wave limit, expression (13) gives

$$\lim_{|k|\to\infty} \omega_\pm = -\frac{2}{9} \pm i|k|\sqrt{3}. \tag{14}$$

As the final comment here, Eq. (14) demonstrates a tendency of the damping rate, $\Re\omega_\pm$, to a finite value, $-\frac{2}{9} \approx -0.22$, as $|k| \to \infty$. This asymptotics is in complete agreement with the data for the hydrodynamic branch of the spectrum of the originating Eq. (1), investigated numerically in [6] (see also Ref. [7]). We conclude with a discussion.

(1) To our knowledge, the example considered above gives the first opportunity to treat the problem of an extension of hydrodynamics into a highly nonequilibrium domain (into a short-wave domain here) on a basis of exact solutions. Exact dispersion relation (13) in the CE procedure is demonstrated to be stable for all wavelengths [8]. It is also remarkable that the result of the CE procedure has a clear nonpolynomial character [9]. As a conjecture here, resulting hydrodynamics is *essentially* nonlocal in space.

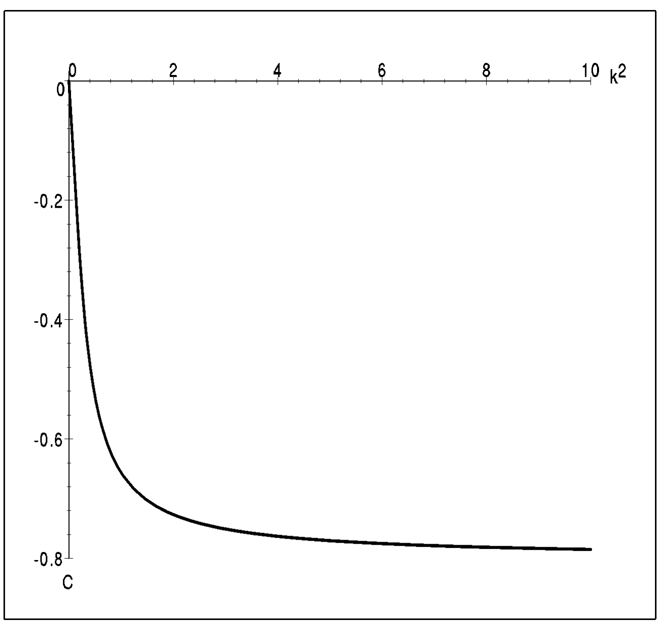

FIG. 1. Real-valued root of Eq. (11) as a function of k^2.

(2) Concerning a derivation of hydrodynamics in a highly nonequilibrium domain from the Boltzmann equations, the situation is, perhaps, not too promising in the sense of an exact summation as above. In this respect, exact results can serve for either a test of approximate procedures or at least for a guide. It is rather remarkable that earlier results on a "partial summing" of recurrence relations such as (3) performed for various Grad moment systems in [4,6] are in qualitative agreement with the result of this Letter. In particular, as applied to the simplest model (1) [4], this approximate method leads to the short-wave asymptotics of the form (14) with other values of constants, and it amounts, in fact, to an approximation of the function $C(k^2)$ with a rational function like $-k^2(1 + \alpha k^2)^{-1}$.

Moreover, a quite different approach [10] which derives hydrodynamics from the Boltzmann equation avoiding the route of expansions in powers of Knudsen number leads again to a nonlocal hydrodynamics (in the linearized one-dimensional case) with features similar to those demonstrated above. The result of this Letter demonstrates that, at least in simple cases, the *sum* of the CE series amounts to a quite regular function, and the "smallness" of Knudsen number ϵ used to develop the CE procedure (3) *is no longer necessary at the outcome*. This final comment encourages a search for approximate procedures, different from the CE original method and not based *a priori* on a small parameter expansion, and further work on relationships between different approaches to hydrodynamics in a highly nonequilibrium domain.

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[7] Besides the hydrodynamic branch with asymptotics (14), the spectrum $\omega(k)$ of Eq. (1) contains a single non-hydrodynamic branch, ω_{nh}, with $\Im\omega_{\mathrm{nh}} = 0$, and with $\Re\omega_{\mathrm{nh}}(0) = -1$, $\Re\omega_{\mathrm{nh}} \to -0.5$, as $|k| \to \infty$.

[8] It remains, however, to be shown that the relevant root of Eq. (11) is a real-valued analytic function of k. This important but technical question amounts to a justification of handling formal expansions, and is not discussed in this Letter.

[9] This follows directly from (12): function $C(k^2)$ cannot be a polynomial because it maps the axis k into a segment $[0, -0.8]$.

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