Hydrodynamics from Grad's equations: What can we learn from exact solutions?

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Abstract

A detailed treatment of the classical Chapman-Enskog derivation of hydrodynamics is given in the framework of Grad's moment equations. Grad's systems are considered as the minimal kinetic models where the Chapman-Enskog method can be studied exactly, thereby providing the basis to compare various approximations in extending the hydrodynamic description beyond the Navier-Stokes approximation. Various techniques, such as the method of partial summation, Padé approximants, and invariance principle are compared both in linear and nonlinear situations.

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Contents

1 Introduction

1.1 The "ultra-violet catastrophe" of the Chapman-Enskog expansion

Most of the interesting expansions in non-equilibrium statistical physics are divergent. This paraphrase of the well known folklore "Dorfman's theorem" conveys the intrinsic problem of many-body systems: A number of systematic (at the first glance) methods has led to (a). An excellent but already known on the phenomenological grounds first approximation. (b). Already the next correction, not known phenomenologically and hence of interest, does not exist because of a divergence. There are many examples of this situations: Cluster expansion of the exact collision integral for dense gases lead to divergent approximations of transport coefficients, non-convergent long tails of correlation functions in the Green-Kubo formulae etc.

Derivation of the hydrodynamic equations from a microscopic description is the classical problem of physical kinetics. As is well known, the famous Chapman-Enskog method [1] provides an opportunity to compute a solution from the Boltzmann kinetic equation as a formal series in powers of the Knudsen number ϵ. The parameter ϵ reflects the ratio between the mean free path of a particle, and the scale of variations of the hydrodynamic fields, the density, the mean flux, and the temperature. If the Chapman-Enskog expansion is truncated at a certain order, we obtain subsequently: the Euler hydrodynamics (ϵ^0), the Navier-Stokes hydrodynamics (ϵ^1), the Burnett hydrodynamics (ϵ^2), the super-Burnett hydrodynamics (ϵ^3), etc. The post-Navier-Stokes terms extend the hydrodynamic description beyond the strictly hydrodynamic limit $\epsilon \ll 1$.

However, as it has been first demonstrated by Bobylev [2], even in the simplest regime (one-dimensional linear deviations around the global equilibrium), the Burnett hydrodynamic equations violate the basic physics behind the Boltzmann equation. Namely, sufficiently short acoustic waves are increasing with time instead of decaying. This contradicts the H-theorem, since all near-equilibrium perturbations must decay. The situation does not improve in the next, super-Burnett approximation.

This "ultra-violet catastrophe" which occurs in the lower-order truncations of the Chapman-Enskog expansion creates therefore very serious difficulties in the problem of an extension of the hydrodynamic description into a highly non-equilibrium domain (see [3] for a discussion of other difficulties of the post-Navier–Stokes terms of the Chapman-Enskog expansion). The

Euler and the Navier–Stokes approximations remain basic in the hydrodynamic description, while the problem of their extension is one of the central open problems of the kinetic theory. The study of approximate solutions based on the Chapman-Enskog method still continues [4].

All this begs for a question: *What is wrong with the Chapman-Enskog method?* At the first glance, the failure of the Burnett and of the super-Burnett hydrodynamics may be accounted in favor of a frequently used argumentation about the asymptotic character of the Chapman-Enskog expansion. However, it is worthwhile to notice here that divergences in the low-order terms of formal expansions are not too surprising. In many occasions, in particular, in quantum field theory [5] and in statistical physics [6], the situation is often improved if one takes into account the very remote terms of corresponding expansions. Thus, a more constructive viewpoint on the Chapman-Enskog expansion could be to proceed along these lines, and to try to *sum up* the Chapman-Enskog series, at least formally and approximately.

An attempt of this kind of working with the Chapman-Enskog expansion is undertaken in this paper. The formalities are known to be rather awkward for the Boltzmann equation, and untill now, exact summations of the Chapman-Enskog expansion are known in a very limited number of cases [7]. In this paper, we will concentrate on the Chapman-Enskog method as applied to the well known Grad moment equations [8]. The use of the Grad equations for our purpose brings, of course, considerable technical simplifications as compared to the case of the Boltzmann equation but it does not make the problem trivial. Indeed, the Chapman-Enskog method amounts to a nonlinear recurrence procedure even as applied to the simplest, linearized Grad equations. Moreover, as we will see it soon, the Chapman-Enskog expansion for moment systems inherits Bobylev's instability in the low-order approximations. Still, the advantage of our approach is that many explicit results can be obtained and analyzed. In order to summarize, in this paper we consider Grad's moment equations as finitely-coupled kinetic models where the problem of reduced description is meaningful, rather than as models of extended hydrodynamics. The latter viewpoint is well known as a microscopic background of the extended irreversible thermodynamics [9, 10].

The outline of this paper is as follows: after an introduction of the Chapman-Enskog procedure for the linearized Grad equations (section 1.2), we will open the discussion with two examples (the linearized one- and three-dimensional 10 moment Grad equations) where the Chapman-Enskog series is summed up exactly in a closed form (sections 2.1 and 2.2). These results

makes it possible to discuss the features of the Chapman-Enskog solution in the short-wave domain in the frames of the model, and will serve for a purpose of testing various approximate methods thereafter. We will see, in particular, that the "smallness" of the Knudsen number ϵ used to develop the Chapman-Enskog method has no direct meaning in the exact result. Also, it will become clear that the finite-order truncations, even provided they are stable, give less opportunities to approximate the solution in a whole, and especially in the short-wave domain.

The exact solutions are, of course, the lucky exceptions, and even for the Grad moment equations the complexity of the Chapman-Enskog method increases rapidly with an increase of the number of the moments taken into account. Further (section 3.1) we will review a technique of summing the Chapman-Enskog expansion *partially*. This technique is heuristic (as are the methods of partial summing in general), but still it removes the Bobylev instability, as well as it qualitatively reproduces the features of the exact solutions in the short-wave limit.

The style of working in the sections mentioned so far falls into a paradigm of the Taylor-like expansions into powers of the Knudsen number. This viewpoint on the problem of a derivation of the hydrodynamics will be *altered* beginning with the section 3.2. There we demonstrate that a condition of a *dynamic invariance* which can be realized directly and with no request of the Knudsen number brings us to the same result as the exact summation of the Chapman-Enskog expansion. The Chapman-Enskog method thereafter can be regarded for a *one* possibility to solve the resulting invariance equations. Further, we demonstrate that iterative methods provide a reasonable alternative to the Taylor expansion in this problem. Namely, we show that the Newton method has certain advantages above the Chapman-Enskog method (section 3.3). We also establish a relationship between the method of partial summing and the Newton method.

The material of further sections serves for an illustrative introduction how the pair 'invariance equation + Newton method' can be applied to the problems of kinetic theory. The remaining sections of this chapter are devoted to further examples of this type of working on the level of the Grad equations. In sections 3.4 and 3.5 we derive and discuss the invariance equations for the linearized thirteen-moment Grad equations. Section 3.6 is devoted to kinetic equations of the Grad type, arising in the problems of phonon transport in massive solids at low temperatures. In particular, we demonstrate that the onset of the second sound regime of phonon propagation corresponds to a branching point of the exact sum of the relevant

Chapman–Enskog expansion.

In section 3.7 we apply the invariance principle to nonlinear Grad equations. We sum up exactly a *subseries* of the Chapman-Enskog expansion, namely, the dominant contribution in the limit of high average velocities. This type of contribution is therefore important for an extension of the hydrodynamic description into the domain of strong shock waves. We give a relevant analysis of the corresponding invariance equation, and, in particular, discuss the nature of singular points of this equation. A brief discussion concludes this paper. Some of the results presented below were published earlier in [11, 12, 13, 14, 15, 16, 17, 18, 19, 20]

1.2 The Chapman-Enskog method for linearized Grad's equations

In this section, for the sake of completeness, we introduce linearized Grad's equations and the Chapman-Enskog method for them in a form to be used in this paper. Since the Chapman-Enskog method is extensively discussed in a number of books, especially, in the classical monograph [1], our presentation will be brief.

Notations will follow those of the papers [2, 11]. We denote ρ_0, T_0 and $\boldsymbol{u} = 0$ as the fixed equilibrium values of the density, of the temperature and of the averaged velocity (in the appropriate Galilean reference frame), while $\delta\rho$, δT and $\delta\boldsymbol{u}$ are small deviations of the hydrodynamic quantities from their equilibrium values. Grad's moment equations [8] which will appear below, contain the temperature dependent viscosity coefficient $\mu(T)$. It is convenient to write $\mu(T) = \eta(T)T$. The functional form of $\eta(T)$ is dictated by the choice of the model for particle's interaction. In particular, we have $\eta = \text{const}$ for Maxwell's molecules, and $\eta \sim \sqrt{T}$ for hard spheres. Let us introduce the following system of dimensionless variables:

$$\begin{aligned} \boldsymbol{u} &= \frac{\delta\boldsymbol{u}}{\sqrt{k_{\mathrm{B}}T_0/m}},\ \rho = \frac{\delta\rho}{\rho_0},\ T = \frac{\delta T}{T_0}, \\ \boldsymbol{x} &= \frac{\rho_0}{\eta(T_0)\sqrt{k_{\mathrm{B}}T_0/m}}\boldsymbol{x}',\ t = \frac{\rho_0}{\eta(T_0)}t', \end{aligned} \tag{1}$$

where $\boldsymbol{x}'$ are spatial coordinates, and t' is the time. In the sequel, we use the system of units in which Boltzmann's constant k_{B} and the particle's mass m are one. Three-dimensional thirteen moment Grad's equations, linearized near the equilibrium, take the following form when written in terms of the

dimensionless variables (1):

$$\begin{aligned} \partial_t \rho &= -\nabla \cdot \boldsymbol{u}, \qquad (2) \\ \partial_t \boldsymbol{u} &= -\nabla \rho - \nabla T - \nabla \cdot \boldsymbol{\sigma}, \\ \partial_t T &= -\frac{2}{3}(\nabla \cdot \boldsymbol{u} + \nabla \cdot \boldsymbol{q}), \\ \partial_t \boldsymbol{\sigma} &= -\overline{\nabla \boldsymbol{u}} - \frac{2}{5}\overline{\nabla \boldsymbol{q}} - \boldsymbol{\sigma}, \qquad (3) \\ \partial_t \boldsymbol{q} &= -\frac{5}{2}\nabla T - \nabla \cdot \boldsymbol{\sigma} - \frac{2}{3}\boldsymbol{q}. \end{aligned}$$

In these equations, $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ and $\boldsymbol{q}(\boldsymbol{x}, t)$ are dimensionless quantities corresponding to the stress tensor and to the heat flux, respectively. Further, the gradient ∇ stands for the vector of spatial derivatives $\partial/\partial \boldsymbol{x}$. The dot denotes the standard scalar product, while the overline stands for a symmetric traceless dyad. In particular,

$$\overline{\nabla \boldsymbol{u}} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T - \frac{2}{3} I \nabla \cdot \boldsymbol{u},$$

where I is unit matrix.

Grad's equations (2) and (3) is the simplest model of a coupling of the hydrodynamic variables, $\rho(\boldsymbol{x}, t)$, $T(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{x}, t)$, to the non-hydrodynamic variables $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ and $\boldsymbol{q}(\boldsymbol{x}, t)$. The problem of reduced description is to close the first three equations (2), and to get an autonomous system for the hydrodynamic variables alone. In other words, the non-hydrodynamic variables $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ and $\boldsymbol{q}(\boldsymbol{x}, t)$ should be expressed in terms of the variables $\rho(\boldsymbol{x}, t)$, $T(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{x}, t)$. The Chapman-Enskog method, as applied for this purpose to Grad's system (2) and (3), involves the following steps:

First, we introduce a formal parameter ϵ, and write instead of equations (3):

$$\begin{aligned} \partial_t \boldsymbol{\sigma} &= -\overline{\nabla \boldsymbol{u}} - \frac{2}{5}\overline{\nabla \boldsymbol{q}} - \frac{1}{\epsilon}\boldsymbol{\sigma}, \qquad (4) \\ \partial_t \boldsymbol{q} &= -\frac{5}{2}\nabla T - \nabla \cdot \boldsymbol{\sigma} - \frac{2}{3\epsilon}\boldsymbol{q}. \end{aligned}$$

Second, the Chapman-Enskog solution is found as a formal expansions of the stress tensor and of the heat flux vector:

$$\begin{aligned} \boldsymbol{\sigma} &= \sum_{n=0}^{\infty} \epsilon^{n+1} \boldsymbol{\sigma}^{(n)}; \qquad (5) \\ \boldsymbol{q} &= \sum_{n=0}^{\infty} \epsilon^{n+1} \boldsymbol{q}^{(n)}. \end{aligned}$$

Zero-order coefficients, $\boldsymbol{\sigma}^{(0)}$ and $\boldsymbol{q}^{(0)}$ are:

$$\boldsymbol{\sigma}^{(0)} = -\overline{\nabla \boldsymbol{u}}, \quad \boldsymbol{q}^{(0)} = -\frac{15}{4}\nabla T. \tag{6}$$

Coefficients of the order $n \geq 1$ are found from the recurrence procedure:

$$\begin{aligned} \boldsymbol{\sigma}^{(n)} &= -\left\{ \sum_{m=0}^{n-1} \partial_t^{(m)} \boldsymbol{\sigma}^{(n-1-m)} + \frac{2}{5}\overline{\nabla \boldsymbol{q}^{(n-1)}} \right\}, \\ \boldsymbol{q}^{(n)} &= -\frac{3}{2}\left\{ \sum_{m=0}^{n-1} \partial_t^{(m)} \boldsymbol{q}^{(n-1-m)} + \nabla \cdot \boldsymbol{\sigma}^{(n-1)} \right\}, \end{aligned} \tag{7}$$

where $\partial_t^{(m)}$ are recurrently defined *Chapman-Enskog operators*. They act on functions $\rho(\boldsymbol{x}, t)$, $T(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{x}, t)$, and on their spatial derivatives, according to the following rule:

$$\begin{aligned} \partial_t^{(m)} D\rho &= \begin{cases} -D\nabla \cdot \boldsymbol{u} & m = 0 \\ 0 & m \geq 1 \end{cases}; \\ \partial_t^{(m)} DT &= \begin{cases} -\frac{2}{3} D\nabla \cdot \boldsymbol{u} & m = 0 \\ -\frac{2}{3} D\nabla \cdot \boldsymbol{q}^{(m-1)} & m \geq 1 \end{cases}; \\ \partial_t^{(m)} D\boldsymbol{u} &= \begin{cases} -D\nabla(\rho + T) & m = 0 \\ -D\nabla \cdot \boldsymbol{\sigma}^{(m-1)} & m \geq 1 \end{cases}. \end{aligned} \tag{8}$$

Here D is an arbitrary differential operator with constant coefficients.

Given the initial condition (6), the Chapman-Enskog equations (7) and (8) are recurrently solvable. Finally, by terminating the computation at the order $N \geq 0$, we obtain the Nth order approximations to the expansions (5), $\boldsymbol{\sigma}_N$ and $\boldsymbol{q}_N$:

$$\boldsymbol{\sigma}_N = \sum_{n=0}^{N} \epsilon^{n+1} \boldsymbol{\sigma}^{(n)}, \quad \boldsymbol{q}_N = \sum_{n=0}^{N} \epsilon^{n+1} \boldsymbol{q}^{(n)}. \tag{9}$$

Substituting these expressions instead of the functions $\boldsymbol{\sigma}$ and $\boldsymbol{q}$ in the equations (2), we close the latter to give the hydrodynamic equations of the order N. In particular, $N = 0$ results in the Navier-Stokes approximation, $N = 1$ and $N = 2$ give the Burnett and the super-Burnett approximations, respectively, and so on.

Though the 'microscopic' features of Grad's moment equations are, of course, much simpler as compared with the Boltzmann equation, the Chapman-Enskog procedure for them just described is not trivial. Our purpose is to

study explicitly the features of the gradient expansions like (5) in the highly non-equilibrium domain, in particular, to find out to what extend the finite-order truncations (9) approximate the solution, and what kind of alternative strategies to find approximations are possible. In the following, when referring to Grad's equations, we use the notation $mDnM$, where m is the spatial dimension of corresponding fields, and n is the number of these fields. For example, the above system is the $3D13M$ Grad's system.

2 Exact summation of the Chapman–Enskog expansion

2.1 The $1D10M$ Grad equations

In this section, we open our discussion with the exact summation of the Chapman-Enskog series for the simplest Grad's system, the one-dimensional linearized ten-moment equations. Throughout the section we use the hydrodynamic variables $p(x,t) = \rho(x,t) + T(x,t)$ and $u(x,t)$, representing the dimensionless deviations of the pressure and of the average velocity from their equilibrium values. The starting point is the linearized Grad's equations for p, u, and σ, where σ is the dimensionless xx-component of the stress tensor:

$$\begin{aligned} \partial_t p &= -\frac{5}{3}\partial_x u, \\ \partial_t u &= -\partial_x p - \partial_x \sigma, \\ \partial_t \sigma &= -\frac{4}{3}\partial_x u - \frac{1}{\epsilon}\sigma. \end{aligned} \tag{10}$$

The system of equations for three functions is the derivative of the ten-moment Grad's system (see Eq. (38) below). Equations (10) provides the simplest model of a coupling of the hydrodynamic variables, u and p, to the single non-hydrodynamic variable σ, and corresponds to a heat non-conductive case.

Our goal here is to shorten the description, and to get a closed set of equations with respect to variables p and u only. That is, we have to express the function σ in the terms of spatial derivatives of the functions p and of u. The Chapman-Enskog method, as applied to eq. (10) results in the following

series representation:

$$\sigma = \sum_{n=0}^{\infty} \epsilon^{n+1} \sigma^{(n)}. \tag{11}$$

Coefficients $\sigma^{(n)}$ are due to the following recurrence procedure [11]:

$$\sigma^{(n)} = - \sum_{m=0}^{n-1} \partial_t^{(m)} \sigma^{(n-1-m)}, \tag{12}$$

where the Chapman-Enskog operators $\partial_t^{(m)}$ act on p, on u, and on their spatial derivatives as follows:

$$\begin{aligned} \partial_t^{(m)} \partial_x^l u &= \begin{cases} -\partial_x^{l+1} p, & m = 0 \\ -\partial_x^{l+1} \sigma^{(m-1)}, & m \geq 1 \end{cases}, \\ \partial_t^{(m)} \partial_x^l p &= \begin{cases} -\frac{5}{3} \partial_x^{l+1} u, & m = 0 \\ 0, & m \geq 1 \end{cases}. \end{aligned} \tag{13}$$

Here $l \geq 0$ is an arbitrary integer, and $\partial_x^0 = 1$. Finally,

$$\sigma^{(0)} = -\frac{4}{3} \partial_x u, \tag{14}$$

which leads to the Navier–Stokes approximation of the stress tensor: $\sigma_{\mathrm{NS}} = \epsilon \sigma^{(0)}$.

Because of a somewhat involved structure of the recurrence procedure (12) and (13), the Chapman-Enskog method is a nonlinear operation even in the simplest model (10). Moreover, the Bobylev instability is again present.

Indeed, computing the coefficients $\sigma^{(1)}$ and $\sigma^{(2)}$ on the basis of the expressions (12), we obtain:

$$\sigma_{\mathrm{B}} = \epsilon \sigma^{(0)} + \epsilon^2 \sigma^{(1)} = -\frac{4}{3} (\epsilon \partial_x u + \epsilon^2 \partial_x^2 p), \tag{15}$$

and

$$\sigma_{\mathrm{SB}} = \epsilon \sigma^{(0)} + \epsilon^2 \sigma^{(1)} + \epsilon^3 \sigma^{(2)} = -\frac{4}{3} (\epsilon \partial_x u + \epsilon^2 \partial_x^2 p + \frac{1}{3} \epsilon^3 \partial_x^3 u), \tag{16}$$

for the Burnett and the super-Burnett approximations, respectively. Now we can substitute each of the approximations, σ_{NS}, σ_{B}, and σ_{SB} instead of the function σ in the second of the equations of the set (10). Then the equations thus obtained, together with the equation for the density

ρ, will give us the closed systems of the hydrodynamic equations of the Navier-Stokes, of the Burnett, and of the super-Burnett levels. To see the properties of the resulting equations, we compute the dispersion relation for the hydrodynamic modes. Using a new space-time scale, $x' = \epsilon^{-1}x$, and $t' = \epsilon^{-1}t$, and next representing $u = u_k\varphi(x', t')$, and $p = p_k\varphi(x', t')$, where $\varphi(x', t') = \exp(\omega t' + ikx')$, and k is a real-valued wave vector, we obtain the following dispersion relations $\omega(k)$ from the condition of a non-trivial solvability of the corresponding linear system with respect to u_k and p_k:

$$\omega_\pm = -\frac{2}{3}k^2 \pm \frac{1}{3}i|k|\sqrt{4k^2 - 15}, \tag{17}$$

for the Navier-Stokes approximation,

$$\omega_\pm = -\frac{2}{3}k^2 \pm \frac{1}{3}i|k|\sqrt{8k^2 + 15}, \tag{18}$$

for the Burnett approximation (15), and

$$\omega_\pm = \frac{2}{9}k^2(k^2 - 3) \pm \frac{1}{9}i|k|\sqrt{4k^6 - 24k^4 - 72k^2 - 135}, \tag{19}$$

for the super-Burnett approximation (16).

These examples demonstrate that the real part $\mathrm{Re}\ \omega_\pm(k) \leq 0$ for the Navier-Stokes (17) and for the Burnett (18) approximations, for all wave vectors. Thus, these approximations describe attenuating acoustic waves. However, for the super-Burnett approximation, the function $\mathrm{Re}\ \omega_\pm(k)$ (19) becomes positive as soon as $|k| > \sqrt{3}$. That is, the equilibrium point is stable within the Navier-Stokes and the Burnett approximation, and it becomes unstable within the super-Burnett approximation for sufficiently short waves. Similar to the case of the Bobylev instability of the Burnett hydrodynamics for the Boltzmann equation, the latter result contradicts the dissipative properties of the Grad system (10): the spectrum of the full $1D10M$ system (10) is stable for arbitrary k.

Our goal now is to sum up the series (11) in a closed form. Firstly, we will make some preparations.

As was demonstrated in [11], the functions $\sigma^{(n)}$ in eqs. (11) and (12) have the following explicit structure to arbitrary order $n \geq 0$:

$$\begin{aligned} \sigma^{(2n)} &= a_n\partial_x^{2n+1}u, \\ \sigma^{(2n+1)} &= b_n\partial_x^{2(n+1)}p, \end{aligned} \tag{20}$$

where the coefficients a_n and b_n are determined through the recurrence procedure (12), and (13). The Chapman-Enskog procedure (12) and (13) can be represented in terms of the real-valued coefficients a_n and b_n (20). We will do this below.

Knowing the structure (20) of the coefficients of the Chapman-Enskog expansion (11), we can write down its formal sum. It is convenient to use the Fourier variables introduced above which amounts essentially to the change $\epsilon\partial_x \to ik$. Substituting the expression (20) into the Chapman-Enskog series (11), we obtain the following formal expression for the Fourier image of the sum:

$$\sigma_k = ikA(k^2)u_k - k^2B(k^2)p_k, \tag{21}$$

where the functions $A(k^2)$ and $B(k^2)$ are formal power series with the coefficients (20):

$$\begin{aligned} A(k^2) &= \sum_{n=0}^{\infty} a_n(-k^2)^n, \\ B(k^2) &= \sum_{n=0}^{\infty} b_n(-k^2)^n. \end{aligned} \tag{22}$$

Thus, the question of the summation of the Chapman-Enskog series (11) amounts to finding the two functions, $A(k^2)$ and $B(k^2)$ (22). Knowing A and B, we derive a dispersion relation for the hydrodynamic modes:

$$\omega_{\pm} = \frac{k^2A}{2} \pm \frac{|k|}{2}\sqrt{k^2A^2 - \frac{20}{3}(1 - k^2B)}. \tag{23}$$

Now we will concentrate on the problem of a computation of the functions A and B (22) in a closed form. For this purpose, we will first express the Chapman-Enskog procedure (12) and (13) in terms of the coefficients a_n and b_n (20). At the same time, our derivation will constitute a proof of the structure (20).

It is convenient to start with the Fourier representation of the equations (12) and (13). Writing $u = u_k \exp(ikx)$, $p = p_k \exp(ikx)$, and $\sigma = \sigma_k \exp(ikx)$, we obtain the following representation:

$$\begin{aligned} \partial_t^{(m)} u_k &= \begin{cases} -ikp_k, & m = 0 \\ -ik\sigma_k^{(m-1)}, & m \geq 1 \end{cases}, \\ \partial_t^{(m)} p_k &= \begin{cases} -\frac{5}{3}iku_k, & m = 0 \\ 0, & m \geq 1 \end{cases}, \end{aligned} \tag{24}$$

while

$$\sigma_k^{(n)} = -\sum_{m=0}^{n-1} \partial_t^{(m)} \sigma_k^{(n-1-m)}, \tag{25}$$

and

$$\begin{aligned} \sigma_k^{(2n)} &= a_n(-k^2)^n iku_k, \\ \sigma_k^{(2n+1)} &= b_n(-k^2)^n(-k^2)p_k. \end{aligned} \tag{26}$$

The Navier–Stokes and the Burnett approximations give $a_0 = -\frac{4}{3}$, and $b_0 = -\frac{4}{3}$. Thus, the structure (26) is proven for $n = 0$.

The further derivation relies upon induction. Let us assume that the ansatz (26) is proven up to the order n. Computing the coefficient $\sigma_k^{(2(n+1))}$ from the equation (25), we have:

$$\sigma_k^{(2(n+1))} = -\partial_t^{(0)}\sigma_k^{(2n+1)} - \sum_{m=0}^{n} \partial_t^{(2m+1)}\sigma_k^{(2(n-m))} - \sum_{m=1}^{n} \partial_t^{(2m)}\sigma_k^{(2(n-m)+1)}. \tag{27}$$

Due to the assumption of the induction, we can adopt the form of the coefficients $\sigma_k^{(j)}$ (26) in all the terms on the right hand side of the latter expression. On the basis of the expressions (26) and (24), we conclude that each summand in the last sum in (27) is equal to zero. Further, the term $\partial_t^{(0)}\sigma_k^{(2n+1)}$ gives a linear contribution:

$$\partial_t^{(0)}\sigma_k^{(2n+1)} = \partial_t^{(0)} b_n(-k^2)^n(-k^2)p_k = -\frac{5}{3}b_n(-k^2)^{n+1}iku_k,$$

while the terms in the remaining sum contribute nonlinearly:

$$\partial_t^{(2m+1)}\sigma_k^{(2(n-m))} = a_{n-m}(-k^2)^{n-m}ik\partial_t^{(2m+1)}u_k = -a_{n-m}a_m(-k^2)^{n+1}iku_k.$$

Substituting the last two expressions into the equation (27), we see that it has just the same structure as the coefficient $\sigma_k^{(2(n+1))}$ in (26). Thus, we come to the first recurrence equation:

$$a_{n+1} = \frac{5}{3}b_n + \sum_{m=0}^{n} a_{n-m}a_m.$$

Computing the coefficient $\sigma_k^{(2(n+1)+1)}$ by the same pattern, we come to the second recurrence equation, and the Chapman-Enskog procedure (12) and

(13) becomes the following equivalent formulation in terms of the coefficients a_n and b_n (20):

$$\begin{aligned} a_{n+1} &= \frac{5}{3}b_n + \sum_{m=0}^{n} a_{n-m}a_m, \\ b_{n+1} &= a_{n+1} + \sum_{m=0}^{n} a_{n-m}b_m. \end{aligned} \tag{28}$$

The initial condition for this set of equations is dictated by the Navier-Stokes and the Burnett terms:

$$a_0 = -\frac{4}{3}, \quad b_0 = -\frac{4}{3} \tag{29}$$

Our goal now is to compute the functions A and B (22) on the basis of the recurrence equations (28). At this point, it is worthwhile to notice that usual routes of dealing with the recurrence system (28) would be either to truncate it at a certain n, or to calculate all the coefficients explicitly, and next to substitute the result into the power series (22). Both these routes are not successful here. Indeed, retaining the coefficients a_0, b_0, and a_1 gives the super-Burnett approximation (16) which has the Bobylev short-wave instability, and there is no guarantee that the same failure will not occur in the higher-order truncation. On the other hand, a term-by-term computation of the whole set of coefficients a_n and b_n is a quite nontrivial task due to a nonlinearity in the equations (28).

Fortunately, another route is possible. Multiplying both the equations in (28) with $(-k^2)^{n+1}$, and performing a formal summation in n from zero to infinity, we arrive at the following expressions:

$$\begin{aligned} A - a_0 &= -k^2 \left\{ \frac{5}{3}B + \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m}a_m(-k^2)^m \right\}, \\ B - b_0 &= A - a_0 - k^2 \sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m}b_m(-k^2)^m. \end{aligned} \tag{30}$$

Now we notice that

$$\begin{aligned} \lim_{N\to\infty} \sum_{n=0}^{N}\sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m}a_m(-k^2)^m &= A^2, \\ \lim_{N\to\infty} \sum_{n=0}^{N}\sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m}b_m(-k^2)^m &= AB. \end{aligned} \tag{31}$$

Accounting the initial condition (29), we come in (30) to a pair of coupled quadratic equations for the functions A and B:

$$\begin{aligned} A &= -\frac{4}{3} - k^2\left(\frac{5}{3}B + A^2\right), \\ B &= A(1 - k^2 B). \end{aligned} \tag{32}$$

The result (32) concludes essentially the question of the computation of the functions A and B (22). Still, further simplifications are possible. In particular, it is convenient to reduce the consideration to a single function. Resolving the system (32) with respect to the function B, and introducing a new function, $X(k^2) = k^2 B(k^2)$, we come to an equivalent cubic equation:

$$-\frac{5}{3}(X-1)^2\left(X + \frac{4}{5}\right) = \frac{X}{k^2}. \tag{33}$$

Since the functions A and B (22) are real-valued, we are interested only in the real-valued roots of eq. (33).

An elementary analysis of this equation brings the following result: *the real-valued root $X(k^2)$ of the equation (33) is unique and negative for all finite values of the parameter k^2*. Moreover, the function $X(k^2)$ is a monotonic function of k^2. The limiting values are:

$$\lim_{|k|\to 0} X(k^2) = 0, \quad \lim_{|k|\to\infty} X(k^2) = -0.8. \tag{34}$$

The function $X(k^2)$ is plotted in Fig. 1.

Under the circumstances just mentioned, a function under the root in the Eq. (23) is negative for all values of the wave vector k, including the limits, and we come to the following dispersion law:

$$\omega_\pm = \frac{X}{2(1-X)} \pm i\frac{|k|}{2}\sqrt{\frac{5X^2 - 16X + 20}{3}}, \tag{35}$$

where $X = X(k^2)$ is the real-valued root of the equation (33). Since the function $X(k^2)$ is a negative function for all $|k| > 0$, the attenuation rate, $Re\omega_\pm$, is negative for all $|k| > 0$, and the exact acoustic spectrum of the Chapman-Enskog procedure *is stable for arbitrary wave lengths*. In the short-wave limit, expression (35) reads:

$$\lim_{|k|\to\infty} \omega_\pm = -\frac{2}{9} \pm i|k|\sqrt{3}. \tag{36}$$

The characteristic equation of the original Grad equations (10) reads:

$$3\omega^3 + 3\omega^2 + 9k^2\omega + 5k^2 = 0. \tag{37}$$

The two complex-conjugated roots of this equation correspond to the hydrodynamic modes, while the non-hydrodynamic real mode, $\omega_{nh}(k)$, and $\omega_{nh}(0) = -1$, and $\omega_{nh} \to -0.5$, as $|k| \to \infty$. Recall that the non-hydrodynamic modes of the Grad equations are characterized by the common property that for them $\omega(0) \neq 0$. These modes are irrelevant to the Chapman-Enskog method. As the final comment here, the expression (36) demonstrates a tendency of the exact attenuation rate, Re $\omega_{\pm}$, to a finite value, $-\frac{2}{9} \approx -0.22$, as $|k| \to \infty$. This asymptotics is in a complete agreement with the data for the hydrodynamic branch of the spectrum (37) of the original Grad equations (10). The attenuation rates (real parts of the dispersion relations $\omega_{\pm}$ for the Burnett (18), the super-Burnett (19), the exact Chapman-Enskog solution (35), are compared to each other in Fig. 2. In this figure, we also represent the attenuation rates of the hydrodynamic and of the non-hydrodynamic mode of the Grad equations (37). The results of this section lead to the following discussion:

(i) The example considered above gives an opportunity to treat the problem of a *summation* of the Chapman-Enskog expansion. The exact dispersion relation (35) of the Chapman-Enskog procedure is demonstrated to be stable for all wave lengths, while the Bobylev instability is present on the level of the super-Burnett approximation. Moreover, it can be demonstrated that the function X (the real root of the equation (33)) is a real-valued analytic function of the variable k. Thus, the treatment of the formal expansions performed above is justified.

(ii) The exact result of the Chapman-Enskog procedure has a clear non-polynomial character. Indeed, this follows directly from (34): the function $X(k^2)$ cannot be a polynomial because it maps the axis k into a segment $[0, -0.8]$. As a conjecture here, the resulting exact hydrodynamics is *essentially* nonlocal in space. For this reason, even if the hydrodynamic equations of a certain level of the approximation *is* stable, it cannot reproduce the non-polynomial behavior for sufficiently short waves.

(iii) The result of this section demonstrates that, at least in some cases, the sum of the Chapman-Enskog series amounts to a quite regular function, and the "smallness" of the Knudsen number ϵ used to develop the Chapman-Enskog procedure (12) *is no more necessary at the outcome*.

2.2 The $3D10M$ Grad equations

In this section we generalize our considerations of the Chapman-Enskog method the three-dimensional linearized 10-moment Grad equations [8]. The Chapman-Enskog series for the stress tensor, which is again due to a non-linear procedure, will be summed up in a closed form. The method we use follows essentially the one discussed above, though the computations are slightly more extensive. The reason to consider this example is that we would like to know what happens to the diffusive hydrodynamic mode in the short-wave domain.

Throughout the section, we use the variables (1), and p and $\boldsymbol{u}$ are dimensionless deviations of the pressure and of the mean flux from their equilibrium values, respectively. The point of departure is the set of the three-dimensional linearized Grad equations for the variables p, $\boldsymbol{u}$, and $\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is a dimensionless stress tensor:

$$\begin{aligned} \partial_t p &= -\frac{5}{3}\nabla\cdot\boldsymbol{u}, \\ \partial_t \boldsymbol{u} &= -\nabla p - \nabla\cdot\boldsymbol{\sigma}, \\ \partial_t \boldsymbol{\sigma} &= -\overline{\nabla\boldsymbol{u}} - \frac{1}{\epsilon}\boldsymbol{\sigma}. \end{aligned} \tag{38}$$

Equation (38) provides a simple model of a coupling of the hydrodynamic variables, $\boldsymbol{u}$ and p, to the non-hydrodynamic variable $\boldsymbol{\sigma}$. These equations are suitable for an application of the Chapman-Enskog procedure. Therefore, our goal here is not to investigate the properties of eq. (38) as they are, but to reduce the description, and to get a closed set of equations with respect to the variables p and $\boldsymbol{u}$ alone. That is, we have to express $\boldsymbol{\sigma}$ in terms of spatial derivatives of p and of $\boldsymbol{u}$. The Chapman-Enskog method, as applied to eq. (38) results in the following:

$$\boldsymbol{\sigma} = \sum_{n=0}^{\infty} \epsilon^{n+1}\boldsymbol{\sigma}^{(n)}. \tag{39}$$

The coefficients $\boldsymbol{\sigma}^{(n)}$ are due to the following recurrence procedure:

$$\boldsymbol{\sigma}^{(n)} = -\sum_{m=0}^{n-1} \partial_t^{(m)}\boldsymbol{\sigma}^{(n-1-m)}, \tag{40}$$

where the Chapman-Enskog operators $\partial_t^{(m)}$ act on the functions p and $\boldsymbol{u}$,

and on their derivatives, as follows:

$$\begin{aligned}
\partial_t^{(m)} D\boldsymbol{u} &= \begin{cases} -D\nabla p, & m=0 \\ -D\nabla\cdot\boldsymbol{\sigma}^{(m-1)}, & m\geq 1 \end{cases}, \\
\partial_t^{(m)} Dp &= \begin{cases} -\frac{5}{3}D\nabla\cdot\boldsymbol{u}, & m=0 \\ 0, & m\geq 1 \end{cases}.
\end{aligned} \tag{41}$$

Here D is an arbitrary differential operator $D=\prod_{i=1}^{3}\partial_i^{l_i}$, while l_i is an arbitrary integer, and $\partial_i^0=1$. Finally, $\boldsymbol{\sigma}^{(0)}=-\overline{\nabla\boldsymbol{u}}$, which leads to the Navier-Stokes approximation.

Our goal is to sum up the series (39) in a closed form. Firstly, we will make some preparations.

As was demonstrated in [11], $\boldsymbol{\sigma}^{(n)}$ in the equations (39), (40), and (41), have the following explicit structure for arbitrary order $n\geq 0$ (a generalization of the expressions (20) onto the three-dimensional case):

$$\begin{aligned}
\boldsymbol{\sigma}^{(2n)} &= a_n\Delta^n\overline{\nabla\boldsymbol{u}}+b_n\Delta^{n-1}G\nabla\cdot\boldsymbol{u}, \\
\boldsymbol{\sigma}^{(2n+1)} &= c_n\Delta^n Gp,
\end{aligned} \tag{42}$$

where $\Delta=\nabla\cdot\nabla$ is the Laplace operator, and the operator G has the form:

$$G=\nabla\nabla-\frac{1}{3}I\Delta=\frac{1}{2}\overline{\nabla\nabla}. \tag{43}$$

The real-valued and yet unknown coefficients a_n, b_n, and c_n in the equation (42) are due to the recurrence procedure (40), and (41). Knowing the structure of the coefficients of the Chapman-Enskog series (42), we can reformulate the Chapman-Enskog solution in terms of a self-consistent recurrence procedure for the coefficients a_n, b_n, and c_n. Let us consider this derivation in more detail.

The point of departure is the Fourier representation of the recurrence equations (40), (41), and (42). Writing

$$\begin{aligned}
\boldsymbol{u} &= \boldsymbol{u}_k\exp(i\boldsymbol{k}\cdot\boldsymbol{x}), \\
p &= p_k\exp(i\boldsymbol{k}\cdot\boldsymbol{x}), \\
\boldsymbol{\sigma}^{(n)} &= \boldsymbol{\sigma}_k^{(n)}\exp(i\boldsymbol{k}\cdot\boldsymbol{x}),
\end{aligned}$$

and introducing the unity vector $\boldsymbol{e}_k$ directed along $\boldsymbol{k}$ ($\boldsymbol{k}=k\boldsymbol{e}_k$), we come in the equations (40), (41), and (42) to the following:

$$\boldsymbol{\sigma}_k^{(n)}=-\sum_{m=0}^{n-1}\partial_t^{(m)}\boldsymbol{\sigma}_k^{(n-1-m)}, \tag{44}$$

$$\begin{aligned}
\partial_t^{(m)} D_k \boldsymbol{u}_k &= \begin{cases} -D_k i\boldsymbol{k} p_k, & m=0 \\ -D_k i\boldsymbol{k}\cdot\boldsymbol{\sigma}_k^{(m-1)}, & m\geq 1 \end{cases}, \qquad (45)\\
\partial_t^{(m)} D_k p_k &= \begin{cases} -\frac{5}{3} D_k i\boldsymbol{k}\cdot\boldsymbol{u}_k, & m=0 \\ 0, & m\geq 1 \end{cases}.
\end{aligned}$$

where D_k is an arbitrary tensor $D_k = \prod_{s=1}^{3}(ik_s)^{l_s}$, and

$$\begin{aligned}
\boldsymbol{\sigma}_k^{(2n)} &= (-k^2)^n (a_n i\overline{\boldsymbol{k}\boldsymbol{u}} + b_n i\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u})), \qquad (46)\\
\boldsymbol{\sigma}_k^{(2n+1)} &= c_n(-k^2)^{n+1}\boldsymbol{g}_k p_k,
\end{aligned}$$

where

$$\boldsymbol{g}_k = (\boldsymbol{e}_k\boldsymbol{e}_k - \frac{1}{3}I) = \frac{1}{2}\overline{\boldsymbol{e}_k\boldsymbol{e}_k}. \qquad (47)$$

From the form of the Navier–Stokes approximation, $\boldsymbol{\sigma}_k^{(0)}$, it follows that $a_0 = -1$ and $b_0 = 0$, while a direct computation of the Burnett approximation leads to:

$$\boldsymbol{\sigma}_k^{(1)} = \frac{1}{2}k^2\boldsymbol{g}_k p_k. \qquad (48)$$

Thus, we have $c_0 = -\frac{1}{2}$, and with this the ansatz (42) is proven for $n = 0$ in both the even and the odd orders.

The further derivation relies upon induction. Let the structure (46) be proven up to the order n. The computation of the next, $n+1$ order coefficient $\boldsymbol{\sigma}_k^{(2(n+1))}$, involves only the terms of the lower order. From the equation (44) we obtain:

$$\boldsymbol{\sigma}_k^{(2(n+1))} = -\partial_t^{(0)}\boldsymbol{\sigma}_k^{(2n+1)} - \sum_{m=1}^{2n+1}\partial_t^{(m)}\boldsymbol{\sigma}_k^{(2n+1-m)}. \qquad (49)$$

The first term in the right hand side depends linearly on the coefficients c_n:

$$\begin{aligned}
-\partial_t^{(0)}\boldsymbol{\sigma}_k^{(2n+1)} &= -c_n(-k^2)^{n+1}\boldsymbol{g}_k\partial_t^{(0)}p_k \qquad (50)\\
&= \frac{5}{3}c_n(-k^2)^{n+1} i\boldsymbol{g}_k\boldsymbol{k}\cdot\boldsymbol{u}_k.
\end{aligned}$$

The remaining terms on the right hand side of the equation (49) contribute nonlinearly. Splitting the even and the odd orders of the Chapman-Enskog operators $\partial_t^{(m)}$, we rewrite the sum in the equation (49):

$$-\sum_{m=1}^{2n+1}\partial_t^{(m)}\boldsymbol{\sigma}_k^{(2n+1-m)} = -\sum_{l=1}^{n}\partial_t^{(2l)}\boldsymbol{\sigma}_k^{(2(n-l)+1)} - \sum_{l=0}^{n}\partial_t^{(2l+1)}\boldsymbol{\sigma}_k^{(2(n-l))}. \qquad (51)$$

Due to (46) and (45), each term in the first sum is equal to zero, and we are left only with the second sum. We compute:

$$\partial_t^{(2l+1)}\boldsymbol{\sigma}_k^{(2(n-l))} = (-k^2)^{n-l}(a_{n-l}i\overline{\boldsymbol{k}\partial_t^{(2l+1)}\boldsymbol{u}_k} + b_{n-l}i\boldsymbol{g}_k\boldsymbol{k}\cdot\partial_t^{(2l+1)}\boldsymbol{u}_k), \quad (52)$$

while

$$\partial_t^{(2l+1)}\boldsymbol{u}_k = -(-k^2)^{l+1}(a_l\boldsymbol{u}_k + \frac{1}{3}(a_l+2b_l)\boldsymbol{e}_k(\boldsymbol{e}_k\cdot\boldsymbol{u}_k)). \quad (53)$$

In the last expression, the use of the following identities was made:

$$\begin{aligned}\boldsymbol{k}\cdot\overline{\boldsymbol{k}\boldsymbol{u}_k} &= k^2(\boldsymbol{u}_k + \frac{1}{3}\boldsymbol{e}_k(\boldsymbol{e}_k\cdot\boldsymbol{u}_k)), \quad (54)\\ \boldsymbol{k}\cdot\boldsymbol{g}_k &= \frac{2}{3}\boldsymbol{k}.\end{aligned}$$

Substituting the expression (53) into the right hand side of the equation (52), and thereafter substituting the result into the right hand side of the expression (51), we come to the following in the right hand side of the equation (49):

$$\begin{aligned}\boldsymbol{\sigma}_k^{(2(n+1))} &= (-k^2)^{n+1}\left(\sum_{m=0}^{n}a_{n-m}a_m\right)i\overline{\boldsymbol{k}\boldsymbol{u}_k} + (-k^2)^{n+1}\left(\frac{5}{3}c_n + \right. \quad (55)\\ &\left.\sum_{m=0}^{n}\left\{\frac{1}{3}(2a_{n-m}+b_{n-m})(a_m+2b_m) + a_{n-m}b_m\right\}\right)i\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k).\end{aligned}$$

The functional structure of the right hand side of this expression is the same as that of the first of the equations in the set (46), and thus we come to the first recurrence equation:

$$\begin{aligned}a_{n+1}\overline{\boldsymbol{k}\boldsymbol{u}_k} + b_{n+1}\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k) &= \left(\sum_{m=0}^{n}a_{n-m}a_m\right)\overline{\boldsymbol{k}\boldsymbol{u}_k} + \quad (56)\\ &\left(\frac{5}{3}c_n + \sum_{m=0}^{n}\left\{\frac{1}{3}(2a_{n-m}+b_{n-m})(a_m+2b_m) + a_{n-m}b_m\right\}\right)\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k).\end{aligned}$$

Considering in the same way the coefficient $\boldsymbol{\sigma}_k^{(2(n+1)+1)}$, we come to the second recurrence equation,

$$c_{n+1} = 2a_{n+1} + b_{n+1} + \frac{2}{3}\sum_{m=0}^{n}(2a_{n-m}+b_{n-m})c_m. \quad (57)$$

Thus, the complete set of the recurrence equations is given by Eq. (56) and Eq. (57). Equation (56) is equivalent to a pair of scalar equations. Indeed, introducing new variables,

$$\begin{aligned} r_n &= \frac{2}{3}c_n, \\ q_n &= \frac{2}{3}(2a_n + b_n), \end{aligned} \tag{58}$$

and using the identity,

$$\overline{\boldsymbol{k}\boldsymbol{u}_k} = (\overline{\boldsymbol{k}\boldsymbol{u}_k} - 2\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)) + 2\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k),$$

and also noticing that

$$\boldsymbol{g}_k : (\overline{\boldsymbol{k}\boldsymbol{u}_k} - 2\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)) = 0,$$

where : denotes the double contraction of tensors, we arrive in the equations (56) and (57) at the following three scalar recurrence relations in terms the coefficients r_n, q_n, and a_n:

$$\begin{aligned} r_{n+1} &= q_{n+1} + \sum_{m=0}^{n} q_{n-m}r_m \\ q_{n+1} &= \frac{5}{3}r_n + \sum_{m=0}^{n} q_{n-m}q_m \\ a_{n+1} &= \sum_{m=0}^{n} a_{n-m}a_m \end{aligned} \tag{59}$$

The initial condition for this system is provided by the explicit form of the Navier–Stokes and of the Burnett approximations, and it reads:

$$r_0 = -4/3, \quad q_0 = -4/3, \quad a_0 = -1. \tag{60}$$

The recurrence relations (59) are completely equivalent to the original Chapman-Enskog procedure (40) and (41). In the one-dimensional case, the recurrence system (59) reduces to the first two equations for r_n and q_n. In this case, the system of recurrence equations is identical (up to the notations) to the recurrence system (28), considered in the preceding section. For what follows, it is important to notice that the recurrence equation for the coefficients a_n is decoupled from the equations for the coefficients r_n and q_n.

Now we will express the Chapman-Enskog series of the stress tensor (39) in terms of the coefficients r_n, q_n, and a_n. Using again the Fourier transform, and substituting the expression (42) into the right hand side of the equation (39), we derive:

$$\boldsymbol{\sigma}_k = A(k^2)(\overline{\boldsymbol{k}\boldsymbol{u}_k} - 2\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k)) + \frac{3}{2}Q(k^2)\boldsymbol{g}_k(\boldsymbol{k}\cdot\boldsymbol{u}_k) - \frac{3}{2}k^2R(k^2)\boldsymbol{g}_k p_k, \quad (61)$$

From here on, we use a new spatial scale which amounts to $\boldsymbol{k}' = \epsilon\boldsymbol{k}$, and drop the prime. The functions $A(k^2)$, $Q(k^2)$, and $R(k^2)$ in the equation (61) are defined by the power series with the coefficients due to (59):

$$\begin{aligned} A(k^2) &= \sum_{n=0}^{\infty} a_n(-k^2)^n, \quad (62)\\ Q(k^2) &= \sum_{n=0}^{\infty} q_n(-k^2)^n,\\ R(k^2) &= \sum_{n=0}^{\infty} r_n(-k^2)^n. \end{aligned}$$

Thus, the question of summation of the Chapman-Enskog series (39) amounts to finding the three functions, $A(k^2)$, $Q(k^2)$, and $R(k^2)$ (62) in the three- and two-dimensional cases, or to the two functions, $Q(k^2)$, and $R(k^2)$ in the one-dimensional case.

Now we will focus on a problem of a computation of the functions (62) from the recurrence equations (59). At this point, it is worthwhile to notice again that a truncation at a certain n is not successful. Indeed, already in the one-dimensional case, retaining the coefficients q_0, and r_0, and q_1 gives the super-Burnett approximation (16) which has the short-wave instability for $k^2 > 3$, as it was demonstrated in the preceding section, and there is no guarantee that the same will not occur in a higher-order truncation.

Fortunately, the route of computations introduced in the preceding section works again. Multiplying each of the equations in (62) with $(-k^2)^{n+1}$, and performing a summation in n from zero to infinity, we derive:

$$\begin{aligned} Q - q_0 &= -k^2\left\{\frac{5}{3}R + \sum_{n=0}^{\infty}\sum_{m=0}^{n} q_{n-m}(-k^2)^{n-m}q_m(-k^2)^m\right\}, \quad (63)\\ R - r_0 &= Q - q_0 - k^2\sum_{n=0}^{\infty}\sum_{m=0}^{n} q_{n-m}(-k^2)^{n-m}r_m(-k^2)^m,\\ A - a_0 &= -k^2\sum_{n=0}^{\infty}\sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m}a_m(-k^2)^m. \end{aligned}$$

Now we notice that

$$\begin{aligned}
\lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{n} a_{n-m}(-k^2)^{n-m}a_m(-k^2)^m &= A^2, \qquad (64)\\
\lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{n} q_{n-m}(-k^2)^{n-m}r_m(-k^2)^m &= QR,\\
\lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=0}^{n} q_{n-m}(-k^2)^{n-m}q_m(-k^2)^m &= Q^2.
\end{aligned}$$

Taking into account the initial conditions (60), and also using the expressions (64), we come in the equation (63) to the following three quadratic equations for the functions A, R, and Q:

$$\begin{aligned}
Q &= -\frac{4}{3} - k^2\left(\frac{5}{3}R + Q^2\right), \qquad (65)\\
R &= Q(1-k^2R),\\
A &= -(1+k^2A^2).
\end{aligned}$$

The result (65) concludes essentially the question of computation of functions (62) in a closed form. Still, further simplifications are possible. In particular, it is convenient to reduce a consideration to a single function within the first two equations in the system (65). Introducing a new function, $X(k^2) = k^2R(k^2)$, we come to an equivalent cubic equation:

$$-\frac{5}{3}(X-1)^2(X+\frac{4}{5}) = \frac{X}{k^2}. \qquad (66)$$

This equation coincides with the equation (66) of the previous section. We will also rewrite the third equation in the system (65) using a function $Y(k^2) = k^2A(k^2)$:

$$Y(1+Y) = -k^2. \qquad (67)$$

The functions of our interest (62) can be straightforwardly expressed in terms of the relevant solutions to the equations (66) and (67). Since all the functions in (62) are real-valued functions, we are interested only in the real-valued roots of the algebraic equations (66) and (67).

The relevant analysis of the cubic equation (66) was already done above: the real-valued root $X(k^2)$ is unique and negative for all finite values of

parameter k^2. Limiting values of the function $X(k^2)$ at $k \to 0$ and at $k \to \infty$ are given by the expression (34):

$$\lim_{k\to 0} X(k^2) = 0, \quad \lim_{k\to\infty} X(k^2) = -\frac{4}{5}.$$

The quadratic equation (67) has no real-valued solutions for $k^2 > \frac{1}{4}$, and it has two real-valued solution for each k^2, where $k^2 < \frac{1}{4}$. We denote $k_c = \frac{1}{2}$ as the corresponding critical value of wave vector. For $k = 0$, one of these roots is equal to zero, while the other is equal to one. The asymptotics $Y \to 0$, as $k \to 0$, answers the question which of these two roots of eq. (67) is relevant to the Chapman-Enskog solution, and we derive:

$$Y = \begin{cases} -\frac{1}{2}\left(1 - \sqrt{1-4k^2}\right) & k < k_c \\ \text{none} & k > k_c \end{cases} \tag{68}$$

The function Y (68) is negative for $k \le k_c$.

From now on, X and Y will denote the relevant roots of the equations (66) and (67) just discussed. The Fourier image of the expression $\nabla \cdot \boldsymbol{\sigma}$ follows from (61):

$$i\boldsymbol{k} \cdot \boldsymbol{\sigma}_k = Y((\boldsymbol{e}_k \cdot \boldsymbol{u}_k)\boldsymbol{e}_k - \boldsymbol{u}_k) - \frac{X}{1-X}(\boldsymbol{e}_k \cdot \boldsymbol{u}_k)\boldsymbol{e}_k - iX\boldsymbol{k}p_k. \tag{69}$$

The latter expression contributes to the right-hand side of the second of equations in the Grad system (38) (more specifically, it contributes to the corresponding Fourier transform of this equation). Knowing (69), we can calculate the dispersion $\omega(\boldsymbol{k})$ of the plane waves $\sim \exp\{\omega t + i\boldsymbol{k} \cdot \boldsymbol{x}\}$ which now follows from the exact solution of the Chapman-Enskog procedure. The calculation of the dispersion relation amounts to an evaluation of the determinant of a $(d+1) \times (d+1)$ matrix, and is quite standard (see, e.g. [21]). We therefore reproduce only the final result. The exact dispersion relation of the hydrodynamic modes reads:

$$(\omega - Y)^{d-1}\left(\omega^2 - \frac{X}{1-X}\omega + \frac{5}{3}k^2(1-X)\right) = 0. \tag{70}$$

Here d is the spatial dimension.

From the dispersion relation (70), we easily derive the following classification of the hydrodynamic modes:

(i) For $d = 1$, the spectrum of the hydrodynamic modes is purely acoustic with the dispersion ω_a which is given by the expression (35):

$$\omega_{\rm a} = \frac{X}{2(1-X)} \pm i\frac{k}{2}\sqrt{\frac{5X^2-16X+20}{3}}, \tag{71}$$

where $X = X(k^2)$ is the real-valued root of eq.(66). Since X is a negative function for all $k > 0$, the attenuation rate of the acoustic modes, Re $\omega_{\rm a}$, is negative for all $k > 0$, and the exact acoustic spectrum of the Chapman-Enskog procedure is free of the Bobylev instability for arbitrary wave lengths.

(ii) For $d > 1$, the dispersion of the acoustic modes is given by the equation (71). As follows from the Chapman-Enskog procedure, the diffusion-like (real-valued) mode has the dispersion $\omega_{\rm d}$:

$$\omega_{\rm d} = \begin{cases} -\frac{1}{2}\left(1-\sqrt{1-4k^2}\right) & k < k_{\rm c} \\ \text{none} & k > k_{\rm c} \end{cases} \tag{72}$$

The diffusion mode is $(d-1)$ times degenerated, the corresponding attenuation rate is negative for $k < k_{\rm c}$, and this mode *cannot be extended beyond the critical value* $k_{\rm c} = \frac{1}{2}$ *within the Chapman-Enskog method.*

The reason why this rather remarkable peculiarity of the Chapman-Enskog procedure occurs can be found upon a closer investigation of the spectrum of the underlying Grad moment system (38).

Indeed, in the original system (38), besides the hydrodynamic modes, there exist several non-hydrodynamic modes which are irrelevant to the Chapman-Enskog solution. All these non-hydrodynamic modes are characterized with a property that corresponding dispersion relations $\omega(\boldsymbol{k})$ do not go to zero, as $k \to 0$. In the point $k_{\rm c} = \frac{1}{2}$, the diffusion branch (72) intersects with one of the non-hydrodynamic branches of the equation (38). For the larger values of the wave vector k, these two branches produce a pair of complex conjugated solutions with the real part equal to $-\frac{1}{2}$. Thus, though the spectrum of the original equations (38) continues indeed after $k_{\rm c}$, *the Chapman-Enskog method does not recognize this extension as a part of the hydrodynamic branch.* It is also interesting to notice that if we would accept all the roots of the equation (67), including the complex values for $k > k_{\rm c}$, and not only the real-valued root as was suggested by the asymptotics of the Chapman-Enskog solution (see the explanations preceding the equation (68)), then we would come in (70) to the structure of the dispersion relation just mentioned.

The attenuation rates (the functions Re $\omega_{\rm a}$ and Re $\omega_{\rm d}$) are represented in Fig. 3, together with the relevant dependencies for the approximations of the Chapman-Enskog method. The non-hydrodynamic branch of eq. (38)

which causes the breakdown of the Chapman-Enskog solution is also represented in Fig. 3. It is rather remarkable that while the exact hydrodynamic description becomes inapplicable for the diffusion branch at $k \geq k_{\rm c}$, the usual Navier–Stokes description is still providing a good approximation to the acoustic mode around this point.

The analysis of this section leads to the following additional remarks to the conclusions made in the end of the section 2.1.:

(i) The example considered above gives an opportunity to discuss the features of Chapman-Enskog solutions and the problem of an extension of the hydrodynamic modes into a highly non-equilibrium domain on the exact basis and in the full spatial dimension. The exact acoustic mode in the framework of the Chapman-Enskog procedure is demonstrated to be stable for all wave lengths, while the diffusion-like mode can be regarded for the hydrodynamic mode only in a bounded domain $k < k_{\rm c}$. It is remarkable that the result of the Chapman-Enskog procedure has a clear non-polynomial character. As a conjecture here, the resulting hydrodynamics is *essentially* nonlocal in space. It is also clear that *any* polynomial approximation to the Chapman-Enskog series will fail to reproduce the peculiarity of the diffusion mode demonstrated in frames of the exact solution.

(ii) Concerning an extension of hydrodynamics into a highly non-equilibrium domain on the basis of the Boltzmann equations, this question remains open in a sense of an exact summation as above. In this respect, results for simplified models can serve either for a test of approximate procedures or at least for a guide. In particular, the mechanism of the singularity of the diffusion-like mode through a coupling to the non-hydrodynamic mode might be a rather general mechanism of limitation of the hydrodynamic description, and not just a feature of the Grad systems.

(iii) The result of this section demonstrates that the sum of the Chapman-Enskog series amounts to either a quite regular function (as is the function X), or to a function with a singularity at finite $k_{\rm c}$. In both cases, however, the "smallness" of the Knudsen number ϵ used to develop the Chapman-Enskog procedure plays no role in the outcome of the Chapman-Enskog procedure.

3 The dynamic invariance principle

3.1 Partial summation of the Chapman-Enskog expansion

The examples considered above demonstrate that it makes sense to speak about the sum of the Chapman-Enskog expansion, at least when the Chapman-Enskog method is applied to the (linearized) Grad equations. However, even in this case, the possibility to perform the summation exactly seems to be a lucky exceptions rather than a rule. Indeed, computations become more bulky with the increase of the number of the moments included in the Grad equations. Therefore, we arrive at a question: how can we approximate the recurrence equations of the Chapman-Enskog method to account all the orders in the Knudsen number? Any such method amounts to some "partial" summation of the Chapman-Enskog expansion, and this type of working with formal series is widely spread in various fields of physics.

In this section we will discuss a method of approximating the Chapman-Enskog expansion in a whole. As we now have the exact expressions for the Chapman-Enskog solution for the linearized 10 moment Grad equations, it is natural to start with this example for the reason of comparison.

Let us come back to the originating one-dimensional Grad equations (10), and to the corresponding formulas of the Chapman-Enskog method (12) and (13). Instead of using the exact equations (12) in each order n, we introduce the following approximate equations:

Let $N \geq 1$ is some fixed integer. Then, instead of equations (12), we write:

$$\sigma^{(n)} = -\sum_{m=0}^{n-1} \partial_t^{(m)} \sigma^{(n-1-m)}, \quad n \leq N, \tag{73}$$

$$\sigma^{(n)} = -\sum_{m=0}^{N-1} \partial_t^{(m)} \sigma^{(n-1-m)}, \quad n > N. \tag{74}$$

This approximation amounts to the following: up to the order N, the Chapman-Enskog procedure (12) is taken exactly (equation (73)), while in the computations of the higher orders (equation (74)) we restrict the set of the Chapman-Enskog operators (13) only to the order N. Thus, the Chapman-Enskog coefficients $\sigma^{(n)}$ of the order higher than N are taken into account only "partially". As N tends to infinity, the recurrence procedure (73) and (74) tends formally to the exact Chapman-Enskog procedure (12). We will further refer to the equations (73) and (74) as to the *regularization* of the $N-th$ order. In particular, taking $N = 1$, we come to the regularization of the Burnett approximation, taking $N = 2$ we come to the regularization of the super-Burnett approximation, etc.

It can be demonstrated that the approximate procedure just described does not alter the structure of the functions $\sigma^{(2n)}$ and $\sigma^{(2n+1)}$ (20), while the recurrence equations for the coefficients a_n and b_n (20) will differ from the exact result of the full Chapman-Enskog procedure (28). The advantage of the regularization procedure (73) and (74) above the exact Chapman-Enskog recurrence procedure (12) is that the resulting equations for the coefficients a_n and b_n are always linear, as they appear from the equations (73) and (74). This feature enables one to sum up the corresponding series exactly, even if the originating nonlinear procedure leads to a too difficult analysis. The number N can be called the "depth" of the approximation: the greater is N the more low-order terms of the Chapman-Enskog expansion are taken into account exactly due to the equations (73).

For the first example, let us take $N = 1$ in the equations (73) and (74). The regularization of the Burnett approximation in accord with these equations reads:

$$\sigma^{(n)} = -\partial_t^{(0)} \sigma^{(n-1)}, \tag{75}$$

where $n \geq 1$, and $\sigma^{(0)} = -(4/3)\partial_x u$. Turning to the Fourier variables, we derive:

$$\begin{aligned} \sigma_k^{(2n)} &= a_n(-k^2)^n iku_k, \\ \sigma_k^{(2n+1)} &= b_n(-k^2)^{n+1} p_k, \end{aligned} \tag{76}$$

while the coefficients a_n and b_n are due to the following recurrence procedure:

$$a_{n+1} = \frac{5}{3} b_n, \quad b_n = a_n, \quad a_0 = -\frac{4}{3}, \tag{77}$$

and whereupon

$$a_n = b_n = \left(\frac{5}{3}\right)^n a_0. \tag{78}$$

Thus, denoting as σ_{1k}^R the Fourier transform of the regularized Burnett approximation, we obtain:

$$\sigma_{1k}^R = -\frac{4}{3 + 5k^2} \left(iku_k - k^2 p_k \right). \tag{79}$$

It should be noticed that the recurrence equations (77) can also be obtained from the exact recurrence equations (28) by canceling the nonlinear

terms. Thus, the approximation adopted within the regularization procedure (75) amounts to the following rational approximation of the functions A and B (22):

$$A_1^R = B_1^R = -\frac{4}{3+5k^2}. \tag{80}$$

Substituting the latter expressions instead of the functions A and B in the formula for the dispersion (23), we come to the dispersion relation of the hydrodynamic modes within the regularized Burnett approximation:

$$\omega_\pm = -\frac{2k^2}{3+5k^2} \pm i|k|\sqrt{\frac{75k^2k^2+66k^2+15}{25k^2k^2+30k^2+9}}. \tag{81}$$

The dispersion relation (81) is stable for all wave vectors, and in the short-wave limit we have:

$$\lim_{|k|\to\infty} \omega_\pm = -0.4 \pm i|k|\sqrt{3}. \tag{82}$$

Thus, the regularized Burnett approximation leads to a qualitatively the same behavior of the dispersion relation, as it is for the exact result (36), giving the limiting value of the attenuation rate equal to -0.4 instead of the exact value $-2/9$.

Consider now the regularization of the super-Burnett approximation. This amounts to setting $N = 2$ in the recurrence equations (73) and (74). Then, instead of the equations (75), we have:

$$\begin{aligned} \sigma^{(1)} &= -\partial_t^{(0)}\sigma^{(0)}, \\ \sigma^{(2+n)} &= -\partial_t^{(0)}\sigma^{(n+1)} - \partial_t^{(1)}\sigma^{(n)}, \end{aligned} \tag{83}$$

where $n \geq 0$. The corresponding recurrence equations for the coefficients a_n and b_n now are as follows:

$$a_{n+1} = \frac{1}{3}b_n, \quad a_n = b_n, \quad a_0 = -\frac{4}{3}. \tag{84}$$

Thus, instead of the expressions (80), we come to the following:

$$A_2^R = B_2^R = -\frac{4}{3+k^2}. \tag{85}$$

The corresponding dispersion relation of the regularized super-Burnett approximation reads:

$$\omega_\pm = -\frac{2k^2}{3+k^2} \pm i|k|\sqrt{\frac{25k^2k^2+78k^2+45}{3k^2k^2+18k^2+27}}, \tag{86}$$

while in the short-wave limit the following asymptotics takes place:

$$\lim_{|k|\to\infty} \omega_{\pm} = -2 \pm i|k|\sqrt{\frac{25}{3}}. \tag{87}$$

The Bobylev instability is removed again within the regularization of the super-Burnett approximation, and also the lower-order terms of the Chapman-Enskog expansion are taken into account more precisely in comparison to the regularized Burnett approximation. However, the approximation in a whole has not improved (see Fig. 4). *Thus, we can conclude that though the partial summation method* (73) *and* (74) *is in a capacity to remove the Bobylev instability, and to reproduce qualitatively the exact Chapman-Enskog solution in the short-wave domain, the exactness, however, does not increase monotonically with the depth of the approximation* N. This drawback of the regularization procedure indicates once again that an attempt to capture the lower-order terms of the Chapman-Enskog procedure does not succeed in a better approximation in a whole.

3.2 The dynamic invariance

All the procedures considered so far (exact or approximate) were taking the Chapman-Enskog expansion as the starting point. However, the result of the summation in these procedures does not involve the Knudsen number ϵ explicitly, neither the sum does apply for a "smallness" of this parameter. Therefore, it makes sense to reformulate the problem of the reduced description (for the Grad equations (10) this amounts to the problem of constructing a function $\sigma_k(u_k, p_k, k)$) in a way where the parameter ϵ does not come into play at all. Further, in a framework of such an approach, we can seek for a method of explicit constructing the function $\sigma_k(u_k, p_k, k)$, and which does not rely upon the Taylor-like expansions as above.

In this section we introduce the approach just mentioned, considering again the illustrative example (10). These ideas will be extensively used in the sequel, and they also constitute the basis of the so-called method of invariant manifold for dissipative systems [23].

Let us rewrite here the equations (10), in the Fourier variables, and canceling the parameter ϵ:

$$\begin{aligned} \partial_t p_k &= -\frac{5}{3} iku_k, \\ \partial_t u_k &= -ikp_k - ik\sigma_k, \end{aligned} \tag{88}$$

$$\partial_t \sigma_k \;\; = \;\; -\frac{4}{3} ik u_k - \sigma_k.$$

The result of the reduction in the system (88) amounts to a function $\sigma_k(u_k, p_k, k)$, which depend parametrically on the hydrodynamic variables u_k and p_k, and also on the wave vector k. Due to the linearity of the problem under the consideration, this function depends linearly on u_k and p_k, and we can start with the form given by the equation (21):

$$\sigma_k(u_k, p_k, k) = ikAu_k - k^2 B p_k, \tag{89}$$

where A and B are undetermined functions of k. Now, however, we do not refer to a power series representation of these functions as in the equations (22).

Given the form of the function $\sigma_k(u_k, p_k, k)$ (89), we can compute its time derivative in *two* different ways. On the one hand, substituting the function (89) into the right hand side of the third of the equations in the set (88), we derive:

$$\partial_t^{\text{micro}} \sigma_k = -ik \left(\frac{4}{3} + A \right) u_k + k^2 B p_k. \tag{90}$$

On the other hand, computing the time derivative due to the first two equations (88), we come to the following:

$$\begin{aligned} \partial_t^{\text{macro}} \sigma_k \;\; &= \;\; \frac{\partial \sigma_k}{\partial u_k} \partial_t u_k + \frac{\partial \sigma_k}{\partial p_k} \partial_t p_k \\ &= \;\; ikA \left(-ikp_k - ik\sigma_k \right) - k^2 B \left(-\frac{5}{3} ik u_k \right) \\ &= \;\; ik \left(\frac{5}{3} k^2 B + k^2 A \right) u_k + k^2 \left(A - k^2 B \right) p_k. \end{aligned} \tag{91}$$

Equating the expressions in the right hand sides of the equations (90) and (91), and requiring that the resulting equality holds for any values of the variables u_k and p_k, we come to the following two algebraic equations:

$$\begin{aligned} F(A, B, k) \;\; &= \;\; -A - \frac{4}{3} - k^2 \left(\frac{5}{3} B + A^2 \right) = 0, \\ G(A, B, k) \;\; &= \;\; -B + A \left(1 - k^2 B \right) = 0. \end{aligned} \tag{92}$$

These equations are nothing else but the equations (32). Recall that equations (32) have been obtained upon the summation of the Chapman-Enskog

expansion. Now, however, we have come to the same result without using the expansion. Thus, the equations (92) (or, equivalently, (32)) can be used as a starting point for the constructing the function (89).

Now it is important to comment on the somewhat formal manipulations which have led to the equations (92). First of all, by the very sense of the reduced description problem, we are looking for a set of functions σ_k which depend on the time only through the time dependence of the hydrodynamic variables u_k and p_k. That is, we are looking for a set (89), which is parameterized with the values of the hydrodynamic variables. Further, the two time derivatives, (90) and (91), are relevant to the "microscopic" and the "macroscopic" evolution within the set (89), respectively. Indeed, the expression in the right hand side of the equation (90) is just the value of the vector field of the original Grad equation in the points of the set (89). On the other hand, the expression (91) reflects the time derivative due to the reduced (macroscopic) dynamics, which, in turn, is self-consistently defined by the form (89). The equations (92) provide, therefore, the *dynamic invariance condition of the reduced description* for the set (89): the function $\sigma_k(u_k(t), p_k(t), k)$ is a solution to both the full Grad system (88) and to the reduced system which consists of the two first (hydrodynamic) equations. For this reason, the equations (92) and their analogs which will be obtained on the similar reasoning, will be called *the invariance equations*.

3.3 The Newton method

Let us concentrate on the problem of solving the invariance equations (92). Clearly, if we are going to develop the functions A and B into the power series (22), we will come back to the Chapman-Enskog procedure. Now, however, we see that the Chapman-Enskog expansion is just a method to solve the invariance equations (92), and maybe not the optimal one.

Another possibility is to use *iterative* methods. Indeed, let us apply the Newton method. The algorithm of is as follows: Let A_0 and B_0 are some initial approximations chosen for the procedure. The correction, $A_1 = A_0 + \delta A_1$ and $B_1 = B_0 + \delta B_1$, due to the Newton iteration is obtained upon a linearization of the equations (92) about the approximation A_0 and B_0. Computing the derivatives, we can represent the equation of the Newton

iteration in the matrix form:

$$\left(\begin{array}{cc} \frac{\partial F(A,B,k)}{\partial A} \mid_{A=A_0,B=B_0} & \frac{\partial F(A,B,k)}{\partial B} \mid_{A=A_0,B=B_0} \\ \frac{\partial G(A,B,k)}{\partial A} \mid_{A=A_0,B=B_0} & \frac{\partial G(A,B,k)}{\partial B} \mid_{A=A_0,B=B_0} \end{array} \right) \left(\begin{array}{c} \delta A_1 \\ \delta B_1 \end{array} \right) + \left(\begin{array}{c} F(A_0,B_0,k) \\ G(A_0,B_0,k) \end{array} \right) = 0. \tag{93}$$

where

$$\begin{aligned} \frac{\partial F(A,B,k)}{\partial A} \mid_{A=A_0,B=B_0} &= -\left(1+2k^2A_0\right), \\ \frac{\partial F(A,B,k)}{\partial B} \mid_{A=A_0,B=B_0} &= -\frac{5}{3}k^2, \\ \frac{\partial G(A,B,k)}{\partial A} \mid_{A=A_0,B=B_0} &= 1-k^2B_0, \\ \frac{\partial G(A,B,k)}{\partial B} \mid_{A=A_0,B=B_0} &= -\left(1+k^2A_0\right). \end{aligned} \tag{94}$$

Solving the system of linear algebraic equations, we come to the first correction δA_1 and δB_1. Further corrections are found iteratively:

$$\begin{aligned} A_{n+1} &= A_n + \delta A_{n+1}, \\ B_{n+1} &= B_n + \delta B_{n+1}, \end{aligned} \tag{95}$$

where $n \geq 0$, and

$$\left(\begin{array}{cc} -(1+2k^2A_n) & -\frac{5}{3}k^2 \\ 1-k^2B_n & -(1+k^2A_n) \end{array} \right) \left(\begin{array}{c} \delta A_{n+1} \\ \delta B_{n+1} \end{array} \right) + \left(\begin{array}{c} F(A_n,B_n,k) \\ G(A_n,B_n,k) \end{array} \right) = 0. \tag{96}$$

Within the algorithm just presented, we come to the problem how to choose the initial approximation A_0 and B_0. Indeed, the method (95) and (96) is applicable formally to any initial approximation, however, the convergence (if at all) might be sensitive to the choice.

For the first experiment let us take the Navier-Stokes approximation of the functions A and B:

$$A_0 = B_0 = -\frac{4}{3}$$

The outcome of the first two Newton iterations (the attenuation rates as they follow from the first and the second Newton iteration) are presented in Fig. 5. It is clearly seen that the Newton iterations converge rapidly to the exact solution for moderate k, however, the asymptotic behavior in the short-wave domain is not improved.

Another possibility is to take the result of the regularization procedure as presented above. Let the regularized Burnett approximation (80) is taken for the initial approximation, that is:

$$A_0 = A_1^R = -\frac{4}{3+5k^2}, \quad B_0 = B_1^R = -\frac{4}{3+5k^2}. \tag{97}$$

Substituting the expression (97) into the equation (95) and (96) for $n = 0$, and after some algebra, we come to the following first correction:

$$\begin{aligned} A_1 &= -\frac{4(27+63k^2+153k^2k^2+125k^2k^2k^2)}{3(3+5k^2)(9+9k^2+67k^2k^2+75k^2k^2k^2)}, \\ B_1 &= -\frac{4(9+33k^2+115k^2k^2+75k^2k^2k^2)}{(3+5k^2)(9+9k^2+67k^2k^2+75k^2k^2k^2)} \end{aligned} \tag{98}$$

The functions (98) are not yet the exact solution to the equations (92) (that is, the functions $F(A_1, B_1, k)$ and $G(A_1, B_1, k)$ are not equal to zero for all k). However, substituting the functions A_1 and B_1 instead of A and B into the dispersion relation (23), we derive in the short-wave limit:

$$\lim_{|k|\to\infty} \omega_\pm = -\frac{2}{9} \pm i|k|\sqrt{3}. \tag{99}$$

That is, already the first Newton iteration, as applied to the regularized Burnett approximation, leads to the exact expression in the short-wave domain. Since the first Newton iteration appears to be asymptotically exact, the next iterations improve the solution only for the intermediate values of k, whereas the asymptotic behaviour remains exact in all the iterations. The attenuation rates for the first and for the second Newton iterations with the initial approximation (97) are represented in Fig. 6. The agreement with the exact solution is excellent.

A one more test is to take the result of the super-Burnett approximation (85) for the initial condition in the Newton procedure (96). As we know, the regularization of the super-Burnett approximation is provides a poorer approximation in comparison to the approximation (97), in particular, in the short-wave domain. Nevertheless, the Newton iterations do converge though less rapidly (see Fig. 7).

The examples considered so far demonstrate that the Newton method, as applied to the invariance equations (92) is a more powerful tool in comparison to the Chapman-Enskog procedure. It is also important that the initial approximation should be "properly chosen", and should reproduce

the features of the solution in a whole (not only in the long-wave limit), at least qualitatively.

The best among the initial approximations considered so far is the regularized Burnett approximation (97). We have already commented on the relation of this approximation to the invariance equations, as well as on its relation to the Chapman-Enskog procedure. The further important observation is as follows:

Let us choose the *Euler* approximation for the functions A and B, that is:

$$A_0 = B_0 = 0 \tag{100}$$

The equation of the first Newton iteration (96) is very simple:

$$\begin{pmatrix} -1 & -\frac{5}{3}k^2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \delta A_1 \\ \delta B_1 \end{pmatrix} + \begin{pmatrix} -\frac{4}{3} \\ 0 \end{pmatrix} = 0, \tag{101}$$

and

$$A_1 = B_1 = -\frac{4}{3+5k^2}. \tag{102}$$

Thus, *the regularized Burnett approximation is at the same time the first Newton correction as applied to the Euler initial approximation.* This propertie distinguishes the regularization of the Burnett approximation among other regularizations. Now the functions (98) can be regarded as the *second* Newton correction as applied to the Euler initial approximation (100).

Finally, let us examine the question what does the Newton method do in a case of singularities. As we have demonstrated in the previous section, the singularity of the diffusion-like mode occurs when this mode couples to a non-hydrodynamic mode of the 10 moment Grad system if the spatial dimension is greater that one.

We leave it here without a proof that the invariance equation method as applied to the 10 moment Grad system (38) leads to the system of equations (65). We have already demonstrated what is the outcome of the Newton method as applied to the first two equations of this system (responsible for the acoustic mode and containing no singularities). The Newton method, as applied to the equation (67), reads:

$$\begin{aligned} Y_{n+1} &= Y_n + \delta Y_{n+1}, \qquad (103) \\ (1+2Y_n)\delta Y_{n+1} + \{Y_n(1+Y_n) + k^2\} &= 0, \end{aligned}$$

where $n \geq 0$, and Y_0 is a chosen initial approximation. Taking the Euler approximation ($Y_0 = 0$), we derive:

$$\begin{aligned} Y_1 &= -k^2, \\ Y_2 &= -\frac{k^2(1+k^2)}{1-2k^2}. \end{aligned} \tag{104}$$

The second approximation, Y_2, is singular at the point $k_2 = \sqrt{1/2}$, and it can be demonstrated that all the further corrections do also have the first singularity at points k_n, while the sequence $k_2, \ldots, k_n$ tends to the actual branching point of the invariance equation (67) $k_{\rm c} = 1/2$. The analysis of further corrections demonstrates that the convergence is very rapid (see Fig. 8).

The approximations (104) demonstrate that unlike the polynomial approximations, the Newton method is sensitive to detect the actual singularities of the hydrodynamic spectrum. Formally, the function Y_2 becomes positive as k becomes larger than k_2, and thus attenuation rate, $\omega_d = Y_2$ becomes positive after this point. However, unlike the super-Burnett approximation for the acoustic mode, this transition occurs now in a singular point. Indeed, the attenuation rate Y_2 tends to "minus infinity", as k tends to k_2 from the left. Thus, as described with the Newton procedure, the non-physical domain is separated from the physical one with an "infinitely viscid" threshold. The occurrence of the poles in the Newton iterations is, of course, quite clear. Indeed, the Newton method involves the derivative of the function $R(Y) = Y(Y+1)$ which appears on the left hand side of the equation (67). The derivative $dR(Y)/dY$ becomes zero in the singularity point $Y_c = -1/2$. The results of this section bring us to the following discussion:

(i) The result of the exact summation of the Chapman-Enskog procedure brings us to the same system of equations as the principle of the dynamic invariance. This is demonstrated above for a specific situation but it holds as well for any (linearized) Grad system. The resulting equations are always *nonlinear* (even for the simplest linearized kinetic systems, such as Grad equations).

(ii) Now we are able to *alter the viewpoint*: the set of the invariance equations can be considered as the basic in the theory, while the Chapman-Enskog method is a way to solve it via an expansion in powers of k. The method of power series expansion is neither the only method to solve equations, nor the optimal. Alternative iteration methods might be better suited to the problem of constructing the reduced description.

(iii) An opportunity to derive the invariance equation in a closed form, and next to solve it this or that way, is, of course, rather exotic. The situation becomes complicated already for the nonlinear Grad equations, and we should not expect anything simple in the case of the Boltzmann equation. Therefore, if we are willing to proceed along these lines in other problems, the attention draws towards the approximate procedures. With this, the question appears: what amount of information is required to execute the procedures? Indeed, the Navier–Stokes approximation can be obtained without any knowledge of the whole nonlinear system of invariance equations. It is important that the Newton method, as applied to our problem, does not require any global information as well. This was demonstrated above by a relation between the first iteration as applied to the Euler approximation and the regularization of the Burnett approximation.

3.4 Invariance equation for the $1D13M$ Grad system

Let us consider as the next example the problem of the reduced description for the one-dimensional thirteen moment Grad system. Using the dimensionless variables as above, we write the one-dimensional version of the Grad equations (2) and (3) in the k-representation:

$$\begin{aligned}
\partial_t \rho_k &= -iku_k, \\
\partial_t u_k &= -ik\rho_k - ikT_k - ik\sigma_k, \\
\partial_t T_k &= -\frac{2}{3}iku_k - \frac{2}{3}ikq_k, \\
\partial_t \sigma_k &= -\frac{4}{3}iku_k - \frac{8}{15}ikq_k - \sigma_k, \\
\partial_t q_k &= -\frac{5}{2}ikT_k - ik\sigma_k - \frac{2}{3}q_k.
\end{aligned} \tag{105}$$

The Grad system (105) provides the simplest coupling of the hydrodynamic variables ρ_k, u_k, and T_k to the non-hydrodynamic variables, σ_k and q_k, the latter corresponds to the heat flux. As above, our goal is to reduce the description for the Grad system (105) to the three hydrodynamic equations with respect to the variables ρ_k, u_k, and T_k. That is, we have to express the functions σ_k and q_k in terms of ρ_k, u_k, and T_k:

$$\begin{aligned}
\sigma_k &= \sigma_k(\rho_k, u_k, T_k, k), \\
q_k &= q_k(\rho_k, u_k, T_k, k).
\end{aligned}$$

The Chapman-Enskog method, as applied for this purpose, results in the following algebraic scheme (we omit the Knudsen number ϵ):

$$\begin{aligned} \sigma_k^{(n)} &= -\left\{ \sum_{m=0}^{n-1} \partial_t^{(m)} \sigma_k^{(n-1-m)} + \frac{8}{15} ikq_k^{(n-1)} \right\} \\ q_k^{(n)} &= -\left\{ \sum_{m=0}^{n-1} \partial_t^{(m)} q_k^{(n-1-m)} + ik\sigma_k^{(n-1)} \right\}, \end{aligned} \tag{106}$$

where the Chapman-Enskog operators act as follows:

$$\begin{aligned} \partial_t^{(m)} \rho_k &= \begin{cases} -iku_k & m = 0 \\ 0, & m \geq 1 \end{cases}, \\ \partial_t^{(m)} u_k &= \begin{cases} -ik(\rho_k + T_k) & m = 0 \\ -ik\sigma_k^{(m-1)}, & m \geq 1 \end{cases}, \\ \partial_t^{(m)} T_k &= \begin{cases} -\frac{2}{3} iku_k & m = 0 \\ -\frac{2}{3} ikq_k^{(m-1)}, & m \geq 1 \end{cases}. \end{aligned} \tag{107}$$

The initial condition for the recurrence procedure (106) reads: $\sigma_k^{(0)} = -\frac{4}{3} iku_k$, and $q_k^{(0)} = -\frac{15}{4} ikT_k$, which leads to the Navier-Stokes-Fourier hydrodynamic equations.

Computing the coefficients $\sigma_k^{(1)}$ and $q_k^{(1)}$, we come to the Burnett approximation:

$$\begin{aligned} \sigma_{1k} &= -\frac{4}{3} iku_k + \frac{4}{3} k^2 \rho_k - \frac{2}{3} k^2 T_k, \\ q_{1k} &= -\frac{15}{4} ikT_k + \frac{7}{4} k^2 u_k. \end{aligned} \tag{108}$$

The Burnett approximation (108) coincides with that obtained from the Boltzmann equation, and it is precisely the case where the instability was first demonstrated in the paper [2].

The structure of the terms $\sigma_k^{(n)}$ and $q_k^{(n)}$ (an analog of the equations (20) and (42)) is as follows:

$$\begin{aligned} \sigma_k^{(2n)} &= a_n (-k^2)^n iku_k, \\ \sigma_k^{(2n+1)} &= b_n (-k^2)^{n+1} \rho_n + c_n (-k^2)^{n+1} T_k, \\ q_k^{(2n)} &= \beta_n (-k^2)^n ik\rho_k + \gamma_n (-k^2)^n iki T_k, \\ q_k^{(2n+1)} &= \alpha_n (-k^2)^{n+1} u_k. \end{aligned} \tag{109}$$

A derivation of the invariance equation for the system (105) goes along the same lines as in the previous section. We seek the functions of the reduced description in the form:

$$\begin{aligned} \sigma_k &= ikAu_k - k^2B\rho_k - k^2CT_k, \\ q_k &= ikX\rho_k + ikYT_k - k^2Zu_k, \end{aligned} \tag{110}$$

where the functions $A, \dots Z$ are a subject of a further analysis.

The invariance condition results in a closed system of equations for the functions A, B, C, X, Y, and Z. As above, computing the microscopic time derivative of the functions (110), due to the two last equations of the Grad system (105) we derive:

$$\begin{aligned} \partial_t^{\text{micro}}\sigma_k &= -ik\left(\frac{4}{3} - \frac{8}{15}k^2Z + A\right)u_k \\ &\quad + k^2\left(\frac{8}{15}X + B\right)\rho_k + k^2\left(\frac{8}{15}Y + C\right)T_k, \\ \partial_t^{\text{micro}}q_k &= k^2\left(A + \frac{2}{3}Z\right)u_k + ik\left(k^2B - \frac{2}{3}X\right)\rho_k - ik\left(\frac{5}{2} - k^2C - \frac{2}{3}Y\right)T_k. \end{aligned} \tag{111}$$

On the other hand, computing the macroscopic time derivative due to the first three equations of the system (105), we obtain:

$$\begin{aligned} \partial_t^{\text{macro}}\sigma_k &= \frac{\partial\sigma_k}{\partial u_k}\partial_t u_k + \frac{\partial\sigma_k}{\partial\rho_k}\partial_t\rho + \frac{\partial\sigma_k}{\partial T_k}\partial_t T_k \\ &= ik\left(k^2A^2 + k^2B + \frac{2}{3}k^2C - \frac{2}{3}k^2k^2CZ\right)u_k \\ &\quad + \left(k^2A - k^2k^2AB - \frac{2}{3}k^2k^2CX\right)\rho_k \\ &\quad + \left(k^2A - k^2k^2AC - \frac{2}{3}k^2k^2CY\right)T_k; \\ \partial_t^{\text{macro}}q_k &= \frac{\partial q_k}{\partial u_k}\partial_t u_k + \frac{\partial q_k}{\partial\rho_k}\partial_t\rho u_k + \frac{\partial q_k}{\partial T_k}\partial_t T_k \\ &= \left(-k^2k^2ZA + k^2X + \frac{2}{3}k^2Y - \frac{2}{3}k^2k^2YZ\right)u_k \\ &\quad + ik\left(k^2Z - k^2k^2ZB + \frac{2}{3}k^2YX\right)\rho_k \\ &\quad + ik\left(k^2Z - k^2k^2ZC + \frac{2}{3}k^2Y^2\right)T_k. \end{aligned} \tag{112}$$

Equating the corresponding expressions in the formulas (111) and (112), we come to the following system of coupled equations:

$$\begin{aligned}
F_1 &= -\frac{4}{3} + \frac{8}{15}k^2 Z - A - k^2 A^2 - k^2 B - \frac{2}{3}k^2 C + \frac{2}{3}k^2 k^2 CZ = 0, \\
F_2 &= \frac{8}{15}X + B - A + k^2 AB + \frac{2}{3}k^2 CX = 0, \\
F_3 &= \frac{8}{15}Y + C - A + k^2 AC + \frac{2}{3}k^2 CY = 0, \\
F_4 &= A + \frac{2}{3}Z + k^2 ZA - X - \frac{2}{3}Y + \frac{2}{3}k^2 YZ = 0, \\
F_5 &= k^2 B - \frac{2}{3}X - k^2 Z + k^2 k^2 ZB - \frac{2}{3}k^2 YX = 0, \\
F_6 &= -\frac{5}{2} + k^2 C - \frac{2}{3}Y - k^2 Z + k^2 k^2 ZC - \frac{2}{3}k^2 Y^2 = 0.
\end{aligned} \tag{113}$$

As above, the invariance equations (113) can be also obtained upon the summation of the Chapman-Enskog expansion, after the Chapman-Enskog procedure is casted into a recurrence relations for the coefficients $a_n, \ldots, \alpha_n$ (109). This route is less straightforward than the one just presented, and we omit the proof.

The Newton method, as applied to the system (113), results in the following algorithm:

Denote as $\boldsymbol{A}$ the six-component vector function $\boldsymbol{A} = (A, B, C, X, Y, Z)$. Let $\boldsymbol{A}_0$ is the initial approximation, then:

$$\boldsymbol{A}_{n+1} = \boldsymbol{A}_n + \delta \boldsymbol{A}_{n+1}, \tag{114}$$

where $n \geq 0$, and the vector function $\delta \boldsymbol{A}_{n+1}$ is a solution to the linear system of equations:

$$\boldsymbol{N}_n \delta \boldsymbol{A}_{n+1} + \boldsymbol{F}_n = 0. \tag{115}$$

Here $\boldsymbol{F}_n$ is the vector function with the components $F_i(\boldsymbol{A}_n)$, and $\boldsymbol{N}_n$ is a 6×6 matrix:

$$\begin{pmatrix}
-(1 + 2k^2 A_n) & -k^2 & -2/3k^2(1 - k^2 Z_n) \\
k^2 B_n - 1 & 1 + k^2 & 2/3k^2 X_n \\
k^2 C_n - 1 & 0 & 1 + 2/3k^2 Y_n + k^2 A_n \\
1 + k^2 Z_n & 0 & 0 \\
0 & k^2(1 + k^2 Z_n) & 0 \\
0 & 0 & k^2(1 + k^2 Z_n)
\end{pmatrix} \tag{116}$$

$$\left.\begin{array}{ccc} 0 & 0 & 2/3k^2(4/5+k^2C_n) \\ 2/3(4/5+k^2C_n) & 0 & 0 \\ 0 & 2/3(4/5+k^2C_n) & 0 \\ -1 & -2/3(1-k^2Z_n) & 2/3+k^2A_n+2/3k^2Y_n \\ -2/3(1+k^2Y_n) & -2/3k^2X_n & -k^2(1-k^2B_n) \\ 0 & -2/3(1+2k^2Y_n) & -k^2(1-k^2C_n) \end{array}\right)$$

The Euler approximation gives: $A_0 = \ldots = Z_0 = 0$, while $F_1 = -4/3$, $F_6 = -5/2$, and $F_2 = \ldots = F_5 = 0$. The first Newton iteration (115) as applied to this initial approximation, leads again to a simple algebraic problem, and we have finally obtained:

$$\begin{aligned} A_1 &= -20\frac{141k^2+20}{867k^4+2105k^2+300}, \\ B_1 &= -20\frac{459k^2k^2+810k^2+100}{3468k^2k^2k^2+12755k^2k^2+11725k^2+1500}, \\ C_1 &= -10\frac{51k^2k^2-485k^2-100}{3468k^2k^2k^2+12755k^2k^2+11725k^2+1500}, \\ X_1 &= -\frac{375k^2(21k^2-5)}{2(3468k^2k^2k^2+12755k^2k^2+11725k^2+1500)}, \\ Y_1 &= -\frac{225(394k^2k^2+685k^2+100)}{4(3468k^2k^2k^2+12755k^2k^2+11725k^2+1500)}, \\ Z_1 &= -15\frac{153k^2+35}{867k^4+2105k^2+300}. \end{aligned} \tag{117}$$

Substituting the expression (109) into the first three equations of the Grad system (105), and proceeding to the dispersion relation as above, we derive the latter in terms of the functions $A, \ldots, Z$:

$$\begin{aligned} \omega^3 &- k^2\left(\frac{2}{3}Y+A\right)\omega^2 \\ &+ k^2\left(\frac{5}{3}-\frac{2}{3}k^2Z-\frac{2}{3}k^2C-k^2B+\frac{2}{3}k^2AY+\frac{2}{3}k^2k^2CZ\right)\omega \\ &+ \frac{2}{3}k^2(k^2X-k^2Y+k^2k^2BY-k^2k^2XC)=0. \end{aligned} \tag{118}$$

When the functions $A_1, \ldots, Z_1$ (117) are substituted instead of $A, \ldots, Z$ into the equation (118), the dispersion relation of the first Newton iteration, as applied to the invariance equations (113) with the Euler initial approximation, is obtained. This result coincides with the regularization of the

Burnett approximation, which was considered in [11]. There it was demonstrate that the equilibrium is stable within this approximation for arbitrary wave lengths. The dispersion relation for the Burnett approximation, in turn, is due to the approximation

$$A = -4/3, \quad B = -4/3, \quad C = 2/3, \quad X = 0, \quad Y = -15/4, \quad Z = -7/4,$$

as it follows from a comparison of the expressions (108) and (110). The dispersion relation for the Burnett approximation coincides with the one obtained in [2] from the Boltzmann equation.

3.5 Invariance equation for the $3D13M$ Grad system

The final example to be considered is the 13 moment Grad system in the full spatial dimension, (2) and (3). Let us rewrite here the original system in terms of Fourier variables:

$$\begin{aligned}
\partial_t \rho_k &= -ik\boldsymbol{e}_k \cdot \boldsymbol{u}_k, \\
\partial_t \boldsymbol{u}_k &= -ik\boldsymbol{e}_k \rho_k - ik\boldsymbol{e}_k T_k - ik\boldsymbol{e}_k \cdot \boldsymbol{\sigma}_k, \\
\partial_t T_k &= -\frac{2}{3} ik(\boldsymbol{e}_k \cdot \boldsymbol{u}_k + \boldsymbol{e}_k \cdot \boldsymbol{q}_k), \\
\partial_t \boldsymbol{\sigma}_k &= -ik\overline{\boldsymbol{e}_k \boldsymbol{u}_k} - \frac{2}{5} ik\overline{\boldsymbol{e}_k \boldsymbol{q}_k} - \boldsymbol{\sigma}_k, \\
\partial_t \boldsymbol{q}_k &= -\frac{5}{2} ik\boldsymbol{e}_k T_k - ik\boldsymbol{e}_k \cdot \boldsymbol{\sigma}_k - \frac{2}{3}\boldsymbol{q}_k.
\end{aligned} \tag{119}$$

Here we have represented the wave vector $\boldsymbol{k}$ as $\boldsymbol{k} = k\boldsymbol{e}_k$, and $\boldsymbol{e}_k$ is the unit vector.

The structure of the even and odd Chapman-Enskog coefficients, $\boldsymbol{\sigma}_k^{(n)}$ and $\boldsymbol{q}_k^{(n)}$, turns out to be as follows:

$$\begin{aligned}
\boldsymbol{\sigma}_k^{(2n)} &= (-k^2)^n ik \left\{ a_n(\overline{\boldsymbol{e}_k \boldsymbol{u}_k} - 2\boldsymbol{g}_k(\boldsymbol{e}_k \cdot \boldsymbol{u}_k)) + b_n \boldsymbol{g}_k(\boldsymbol{e}_k \cdot \boldsymbol{u}_k) \right\}, \\
\boldsymbol{\sigma}_k^{(2n+1)} &= (-k^2)^{n+1} \boldsymbol{g}_k \{ c_n T_k + d_n \rho_k \}, \\
\boldsymbol{q}_k^{(2n)} &= (-k^2)^n ik\boldsymbol{e}_k \{ \gamma_n T_k + \delta_n \rho_k \}, \\
\boldsymbol{q}_k^{(2n+1)} &= (-k^2)^{n+1} \{ \alpha_n \boldsymbol{e}_k (\boldsymbol{e}_k \cdot \boldsymbol{u}_k) + \beta_n (\boldsymbol{u}_k - \boldsymbol{e}_k (\boldsymbol{e}_k \cdot \boldsymbol{u}_k)) \},
\end{aligned} \tag{120}$$

where $\boldsymbol{g}_k = 1/2\overline{\boldsymbol{e}_k \boldsymbol{e}_k}$, and the real-valued coefficients $a_n, \ldots, \beta_n$ are due to the Chapman-Enskog procedure (7) and (8).

The expressions just presented prompt that the dynamic invariant form of the stress tensor and of the heat flux reads:

$$\begin{aligned}\boldsymbol{\sigma}_k &= ikA(\overline{\boldsymbol{e}_k\boldsymbol{u}_k} - 2\boldsymbol{g}_k(\boldsymbol{e}_k\cdot\boldsymbol{u}_k)) + 2ikB\boldsymbol{g}_k(\boldsymbol{e}_k\cdot\boldsymbol{u}_k) \\ &\quad -2k^2C\boldsymbol{g}_kT_k - 2k^2D\boldsymbol{g}_k\rho_k, \\ \boldsymbol{q}_k &= ikZ\boldsymbol{e}_kT_k + ikU\boldsymbol{e}_k\rho_k \\ &\quad -k^2X(\boldsymbol{u}_k - \boldsymbol{e}_k(\boldsymbol{e}_k\cdot\boldsymbol{u}_k)) - k^2Y\boldsymbol{e}_k(\boldsymbol{e}_k\cdot\boldsymbol{u}_k),\end{aligned} \tag{121}$$

where the functions $A,\ldots,Y$ depend on $\boldsymbol{k}$. The dynamic invariance condition results in the following two closed systems for these functions:

$$\begin{aligned}\frac{2}{5}U + D - B + \frac{2}{3}k^2CU + \frac{4}{3}k^2BD &= 0, \\ \frac{2}{5}Z + C - B + \frac{2}{3}k^2CZ + \frac{4}{3}k^2BC &= 0, \\ -1 + \frac{2}{5}k^2Y - B - \frac{2}{3}k^2C - k^2D - \frac{4}{3}k^2B^2 + \frac{2}{3}k^2k^2CY &= 0, \\ \frac{4}{3}k^2D - \frac{2}{3}U - k^2Y - \frac{2}{3}k^2ZU + \frac{4}{3}k^2k^2YD &= 0, \\ -\frac{5}{2} + \frac{4}{3}k^2C - \frac{2}{3}Z - k^2Y - \frac{2}{3}k^2Z^2 + \frac{4}{3}k^2k^2YC &= 0, \\ \frac{4}{3}B + \frac{2}{3}Y - U - \frac{2}{3}Z + \frac{2}{3}k^2ZY + \frac{4}{3}k^2YB &= 0,\end{aligned} \tag{122}$$

and

$$\begin{aligned}-1 - A + \frac{2}{5}k^2X - k^2A^2 &= 0, \\ A + \frac{2}{3}X + k^2AX &= 0\end{aligned} \tag{123}$$

The method of summation of the Chapman-Enskog expansion can be also developed, starting with the structure of the Chapman-Enskog coefficients (120), and in the same manner as in section 2. Simple but rather extensive computations in this case lead, of course, to the invariance equations (122) and (123).

The Newton method, as applied to the systems (122) and (123) with the initial Euler approximation, leads in the first iteration to the regularization of the Burnett approximation reported earlier in [11].

Introducing the functions $\bar{A} = k^2A$ and $\bar{X} = k^2X$, we come in the equation (123) to the following:

$$R(\bar{A}) = \frac{5\bar{A}(3\bar{A}^2 + 5\bar{A} + 2)}{4(6\bar{A} + 5)} = -k^2, \tag{124}$$

while

$$\bar{X} = -\frac{3\bar{A}}{2 + 3\bar{A}}.$$

The derivative, $dR(\bar{A})/d\bar{A}$, becomes equal to zero in the point $\bar{A}_c \approx -0.364$, which gives the critical wave vector $k_{\rm c} = \sqrt{-R(\bar{A}_c)} \approx 0.305$. The Newton method, as applied to the equation (124) with the initial Euler condition $\bar{A} = 0$, gives the following: the outcomes of the first and of the second iterations are regular functions, while the third and the further iterations bring a singularity which converges to the point $k_{\rm c}$ (see Fig. 9). These singularities (the real poles) of the Newton corrections are of the same nature as discussed above.

3.6 Gradient expansions in kinetic theory of phonons

3.6.1 Exact Chapman-Enskog solution and onset of second sound

In this section, we close our discussion of linearized Grad systems with an application to simple models of the phonon transport in rigid insulators. It is demonstrated that the extended diffusion mode transforms into second sound mode due to its coupling to a non-hydrodynamic mode at some critical value of the wave vector. This criticality shows up as a branching point of the extension of the diffusion mode within the Chapman–Enskog method. Though the analysis is essentially similar to the examples considered above, it is presented in some details for the sake of completeness.

Experiments on heat pulses propagation through crystalline media [24] confirmed existence of a temperature window (the Guyer-Krumhansl window [25]) with respect to which the features of heat propagation are qualitatively different: At temperatures exceeding the high-temperature edge of the window, the heat propagates in a diffusion-like way. Below the low-temperature edge of the window, the propagation goes in a ballistic way, with a constant speed of sound. Within the window, the propagation becomes wave-like. This latter regime is called second sound (see [26] for a review).

This problem has drawn some renewed attention in the last years. Models relevant for a unified description of diffusion, second sound, and ballistic regimes of heat propagation are intensively discussed (see [27, 28] and references therein). To be specific, recall the simplest and typical model of the phonon transport [27]. Let $e(\boldsymbol{x}, t)$ and $\boldsymbol{p}(\boldsymbol{x}, t)$ be small deviations of

the energy density and of the energy flux of the phonon field from their equilibrium values, respectively. Then

$$\partial_t e = -c^2 \nabla \cdot \boldsymbol{p}, \tag{125}$$

$$\partial_t \boldsymbol{p} = -\frac{1}{3}\nabla e - \frac{1}{\tau_R}\boldsymbol{p}. \tag{126}$$

Here c is the Debye velocity of phonons, and τ_R is the characteristic time of resistive processes. Equations (125) can be derived from the Boltzmann–Peierls kinetic equation, within the relaxation time approximation, by a method similar to the Grad method [27]. Eqs. (125) provide the simplest model of a coupling between the hydrodynamic variable e and the non-hydrodynamic variable $\boldsymbol{p}$, allowing for a qualitative description of both the diffusion and the second sound. Following the standard argumentation [27], we observe the two limiting cases: 1). As $\tau_R \to 0$, the equation (126) yields the Fourier relation $\boldsymbol{p} = -\frac{1}{3}\tau_R \nabla e$ which closes the equation (125) to give the diffusion equation:

$$\partial_t e + \frac{1}{3}\tau_R c^2 \Delta e = 0. \tag{127}$$

2). As $\tau_R \to \infty$, the equation (126) yields $\partial_t \boldsymbol{p} = -\frac{1}{3}\nabla e$, and the equation (125) closes to give the wave equation:

$$\partial_t^2 e + \frac{1}{3}c^2 \Delta e = 0. \tag{128}$$

The equation (127) describes the usual diffusive regime of the heat propagation, while the equation (128) is relevant to the (undamped) second sound regime with the velocity $u_2 = c/\sqrt{3}$, and are both closed with respect to the variable e.

However, even within the simplest model (125), the problem of closure remains unsolved in a systematic way when τ_R is finite. The natural way of doing so is provided by the Chapman-Enskog method. In the situation under consideration, the Chapman-Enskog method yields an extension of the diffusive transport to finite values of the parameter τ_R, and leads to an expansion of the non-hydrodynamic variable $\boldsymbol{p}$ in terms of the hydrodynamic variable e. With this, if we are able to make this extension of the diffusive mode exactly, we could learn more about the transition between the diffusion and second sound (within the frames of the model).

The Chapman-Enskog method, as applied to the model (125), results in the following series representation:

$$\boldsymbol{p} = \sum_{n=0}^{\infty} \boldsymbol{p}^{(n)}, \tag{129}$$

where the coefficients $\boldsymbol{p}^{(n)}$ are due to the Chapman-Enskog recurrence procedure,

$$\boldsymbol{p}^{(n)} = -\tau_R \sum_{m=0}^{n-1} \partial_t^{(m)} \boldsymbol{p}^{(n-1-m)}, \tag{130}$$

while the Chapman-Enskog operators $\partial_t^{(m)}$ act on e as follows:

$$\partial_t^{(m)} e = -c^2 \nabla \cdot \boldsymbol{p}^{(m)}. \tag{131}$$

Finally, the zero order term reads: $\boldsymbol{p}^{(0)} = -\frac{1}{3}\tau_R \nabla e$, and leads to the Fourier approximation of the energy flux.

To sum up the series (129) in a closed form, we, will specify the nonlinearity appearing in equations (130) and (131). The coefficients $\boldsymbol{p}^{(n)}$ in equations (129) and (130) have the following explicit structure for arbitrary order $n \geq 0$:

$$\boldsymbol{p}^{(n)} = a_n \Delta^n \nabla e, \tag{132}$$

where the real-valued and yet unknown coefficients a_n are due to the recurrence procedure (130), and (131). Indeed, the form (132) is true for $n = 0$ ($a_0 = -\frac{1}{3}\tau_R$). Let us assume that (132) is proven up to the order $n-1$. Then, computing the nth order coefficient $\boldsymbol{p}^{(n)}$, we derive:

$$\begin{aligned} \boldsymbol{p}^{(n)} &= -\tau_R \sum_{m=0}^{n-1} \partial_t^{(m)} a_{n-1-m} \Delta^{(n-1-m)} \nabla e \\ &= -\tau_R \sum_{m=0}^{n-1} a_{n-1-m} \Delta^{(n-1-m)} \nabla \left(-c^2 a_m \nabla \cdot \nabla \Delta^m e \right) \\ &= \tau_R c^2 \left\{ \sum_{m=0}^{n-1} a_{n-1-m} a_m \right\} \Delta^n \nabla e. \end{aligned} \tag{133}$$

The last expression has the form (132). Thus, the Chapman-Enskog procedure for the model (125) is equivalent to the following nonlinear recurrence relation in terms of the coefficients a_n:

$$a_n = \tau_R c^2 \sum_{m=0}^{n-1} a_{n-1-m} a_m, \tag{134}$$

subject to the initial condition $a_0 = -\frac{1}{3}\tau_R$. Further, it is convenient to make the Fourier transform. Using $\boldsymbol{p} = \boldsymbol{p}_k \exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}$ and $e = e_k \exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}$, where $\boldsymbol{k}$ is the real-valued wave vector, we derive in (132): $\boldsymbol{p}_k^{(n)} = a_n i\boldsymbol{k}(-k^2)^n e_k$, and

$$\boldsymbol{p}_k = i\boldsymbol{k} A(k^2) e_k, \tag{135}$$

where

$$A(k^2) = \sum_{n=0}^{\infty} a_n (-k^2)^n. \tag{136}$$

Thus, the the Chapman-Enskog solution (129) amounts to finding the function $A(k^2)$ represented by the power series (136). If the function A is known, the exact Chapman-Enskog closure of the system (125) amounts to the following dispersion relation of plane waves $\sim \exp\{\omega_k t + i\boldsymbol{k}\cdot\boldsymbol{x}\}$:

$$\omega_k = c^2 k^2 A(k^2). \tag{137}$$

Here ω_k is a complex-valued function of the real-valued vector $\boldsymbol{k}$: $Re\omega_k$ is the attenuation rate, $Im\omega_k$ is the frequency.

Multiplying both the equations in (134) with $(-k^2)^n$, and performing a summation in n from 1 to infinity, we get:

$$A - a_0 = -\tau_R c^2 k^2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_{n-m} (-k^2)^{n-m} a_m (-k^2)^m,$$

Now we notice that

$$\lim_{N\to\infty} \sum_{n=0}^{N} \sum_{m=0}^{n} a_{n-m} (-k^2)^{n-m} a_m (-k^2)^m = A^2,$$

Accounting $a_0 = -\frac{1}{3}\tau_R$, we come to a quadratic equation for the function A:

$$\tau_R c^2 k^2 A^2 + A + \frac{1}{3}\tau_R = 0. \tag{138}$$

Further, a selection procedure is required to choose the relevant root of the equation (138). Firstly, recall that all the coefficients a_n (132) are real-valued by the sense of the Chapman-Enskog method (130) and (131), hence

the function A (136) is real-valued. Therefore, only the real-valued root of the equation (138) is relevant to the Chapman-Enskog solution. The first observation is that the equation (138) has no real-valued solutions as soon as k is bigger than the critical value $k_{\rm c}$, where

$$k_{\rm c} = \frac{\sqrt{3}}{2\tau_R c}. \tag{139}$$

Secondly, there are two real-valued solutions to the equation (138) at $k < k_{\rm c}$. However, only one of them satisfies the Chapman-Enskog asymptotic $\lim_{k\to 0} A(k^2) = -\frac{1}{3}\tau_R$.

With the two remarks just given, we finally derive the following exact Chapman-Enskog dispersion relation (137):

$$\omega_k = \begin{cases} -(2\tau_R)^{-1}\left(1 - \sqrt{1-(k^2)/(k_{\rm c}^2)}\right) & k < k_{\rm c} \\ \text{none} & k > k_{\rm c} \end{cases}. \tag{140}$$

The Chapman-Enskog dispersion relation corresponds to the extended diffusion transport, and it comes back to the standard Fourier approximation in the limit of long waves $k/k_{\rm c} \ll 1$. The Chapman-Enskog solution does not exist as soon as $k/k_{\rm c} > 1$. In the point $k_{\rm c}$, the extended diffusion branch crosses one of the non-hydrodynamic branches of the (125). For larger k, the extended diffusion mode and the critical non-hydrodynamic mode produce a pair of complex conjugated solutions with the real part equal to $-\frac{1}{2\tau_R}$. The imaginary part of this extension after $k_{\rm c}$ has the asymptotics $\pm iu_2 k$, as $k \to \infty$, and where $u_2 = c/\sqrt{3}$ is the (undamped) second sound velocity in the model (125) (see equation (128)). Though the spectrum of the original eq. (125) continues indeed after $k_{\rm c}$, the Chapman-Enskog method does not recognize this extension as a part of the hydrodynamic branch, *while the second sound regime is born from the extended diffusion after the coupling to the critical non-hydrodynamic mode.*

Finally, let us consider the opportunities provided by the Newton method as applied to the invariance equation. First, the invariance equation can be easily obtained in a closed form here. Consider again the expression for the heat flux in terms of the energy density (135), $\boldsymbol{p}_k = i\boldsymbol{k}A(k^2)e_k$, where now the function A is not thought as the Chapman-Enskog series (136). The invariance equation is a constraint on the function A, expressing the form-invariance of the heat flux (135) under both the dynamic equations (125) and (126). Computing the time derivative of the function (135) due

to equation (125), we obtain:

$$\partial_t^{\text{macro}} \boldsymbol{p}_k = i\boldsymbol{k} A(k^2) \partial_t e_k = c^2 k^2 A^2 i\boldsymbol{k} e_k. \tag{141}$$

On the other hand, computing the time derivative of the same function due to equation (126), we have:

$$\partial_t^{\text{micro}} \boldsymbol{p}_k = -\frac{1}{3} i\boldsymbol{k} e_k - \frac{1}{\tau_R} A i\boldsymbol{k} e_k. \tag{142}$$

Equating the expressions (141) and (142), we come to the desired invariance equation for the function A. This equation coincides with the exact Chapman-Enskog equation (138).

As the second step, let us apply the Newton method to the invariance equation (138), taking the Euler approximation ($A_0^N \equiv 0$) for the initial condition. Rewriting the equation (138) in the form $F(A, k^2) = 0$, we come to the following Newton iterations:

$$\left.\frac{dF(A,k^2)}{dA}\right|_{A=A_n} (A_{n+1} - A_n) + F(A_n, k^2) = 0. \tag{143}$$

The first two iterations give:

$$\tau_R^{-1} A_1 = -\frac{1}{3}, \tag{144}$$

$$\tau_R^{-1} A_2 = -\frac{1 - \frac{1}{4}y^2}{3(1 - \frac{1}{2}y^2)}. \tag{145}$$

The first Newton iteration (144) coincides with the first term of the Chapman-Enskog expansion. The second Newton iteration (145) is a rational function with the Taylor expansion coinciding with the Chapman-Enskog solution up to the super-Burnett term, and it has a pole at $y_2 = \sqrt{2}$. The further Newton iterations are also rational functions with the relevant poles in the points y_n, and the sequence of this points tends very rapidly to the location of the actual singularity $y_c = 1$ ($y_3 \approx 1.17$, $y_4 \approx 1.01$, etc.).

3.6.2 Inclusion of normal processes

The account for normal processes in frames of the semi-hydrodynamical models [27] leads to the following generalization of the Eq. 1 25 (written in Fourier variables, in the one-dimensional case):

$$\partial_t e_k = -ikc^2 p_k, \tag{146}$$

$$\partial_t p_k = -\frac{1}{3}ike_k - ikN_k - \frac{1}{\tau_R}p_k, \tag{147}$$

$$\partial_t N_k = -\frac{4}{15}ikc^2 p_k - \frac{1}{\tau}N_k. \tag{148}$$

Here $\tau = \tau_N\tau_R/(\tau_N + \tau_R)$, τ_N is the characteristic time of normal processes, and N_k is the additional field variable. Following the principle of invariance as explained in the preceding section, we write the closure relation for the non-hydrodynamic variables p_k and N_k as:

$$p_k = ikA_k e_k, \quad N_k = B_k e_k, \tag{149}$$

where A_k and B_k are two unknown functions of the wave vector k. Further, following the principle of invariance as explained in the preceding section, each of the relations (149) should be invariant under the dynamics due to Eq. (146), and due to Eqs. (147) and (148). This results in two equations for the functions A_k and B_k:

$$k^2c^2A_k^2 = -\frac{1}{\tau_R}A_k - B_k - \frac{1}{3}, \tag{150}$$

$$k^2c^2A_kB_k = -\frac{1}{\tau}B_k + \frac{4}{15}k^2c^2A_k.$$

When the energy balance equation (146) is closed with the relation (149), this amounts to a dispersion relation for the extended diffusion mode, $\omega_k = k^2c^2A_k$, where A_k is the solution to the invariance equations (150), *subject to the condition* $A_k \to 0$ as $k \to 0$. Resolving equations (150) with respect to A_k, and introducing $\overline{A}_k = k^2c^2A_k$, we arrive at the following:

$$\Phi(\overline{A}_k) = \frac{5\overline{A}_k(1+\tau\overline{A}_k)(\tau_R\overline{A}_k+1)}{5+9\tau\overline{A}_k} = -\frac{1}{3}\tau_R k^2c^2. \tag{151}$$

The invariance equation (151) is completely analogous to the Eq. (138). Written in the form (151), it allows for a direct investigation of the critical points. For this purpose, we find zeroes of the derivative, $d\Phi(\overline{A}_k)/d\overline{A}_k = 0$. When the roots of the latter equation, $\overline{A}_k^c$, are found, the critical values of the wave vector are given as $-(1/3)k_c^2c^2 = \Phi(\overline{A}_k^c)$. The condition $d\Phi(\overline{A}_k)/d\overline{A}_k = 0$ reads:

$$18\tau^2\tau_R\overline{A}_k^3 + 3\tau(3\tau+8\tau_R)\overline{A}_k^2 + 10(\tau+\tau_R)\overline{A}_k + 5 = 0. \tag{152}$$

Let us consider the particularly interesting case, $\epsilon = \tau_N/\tau_R \ll 1$ (the normal events are less frequent than resistive). Then the real-valued root of the equation (152), $\overline{A}_k(\epsilon)$, corresponds to the coupling of the extended diffusion mode to the critical non-hydrodynamic mode. The corresponding modification of the critical wave vector $k_{\rm c}$ (139) due to the normal processes amounts to a shifts towards shorter waves, and we derive:

$$[k_{\rm c}(\epsilon)]^2 = k_{\rm c}^2 + \frac{3\epsilon}{10\tau_R^2 c^2}. \tag{153}$$

3.6.3 Account for anisotropy

The above examples concerned the isotropic Debye model. Let us consider the simplest anisotropic model of a cubic media with a longitudinal (L) and two degenerated transverse (T) phonon modes, taking into account resistive processes only. Introduce the Fourier variables, e_k, e_k^T, $\boldsymbol{p}_k^T$, and $\boldsymbol{p}_k^L$, where $e_k = e_k^L + 2e_k^T$ is the Fourier transform of the total energy of the three phonon modes (the only conserving quantity), while the rest of variables are specific quantities. The isotropic model (125) generalizes to give [27]:

$$\partial_t e_k = -ic_L^2 \boldsymbol{k}\cdot\boldsymbol{p}_k^L - 2ic_T^2 \boldsymbol{k}\cdot\boldsymbol{p}_k^T, \tag{154}$$

$$\partial_t e_k^T = -ic_T^2 \boldsymbol{k}\cdot\boldsymbol{p}_k^T + \frac{1}{\lambda}\left[c_L^3(e_k - 2e_k^T) - c_T^3 e_k^T\right], \tag{155}$$

$$\partial_t \boldsymbol{p}_k^L = -\frac{1}{3}i\boldsymbol{k}(e_k - 2e_k^T) - \frac{1}{\tau_R^L}\boldsymbol{p}_k^L, \tag{156}$$

$$\partial_t \boldsymbol{p}_k^T = -\frac{1}{3}i\boldsymbol{k}e_k^T - \frac{1}{\tau_R^T}\boldsymbol{p}_k^T, \tag{157}$$

where $\lambda = \tau_R^T c_T^3 + 2\tau_R^L c_L^3$. The term containing the factor λ^{-1} corresponds to the energy exchange between the L and T phonon modes. The invariance constraint for the closure relations,

$$\boldsymbol{p}_k^L = i\boldsymbol{k}A_k e_k, \quad \boldsymbol{p}_k^T = i\boldsymbol{k}B_k e_k, \quad e_k^T = X_k e_k, \tag{158}$$

result in the following invariance equations for the $\boldsymbol{k}$-dependent functions A_k, B_k, and X_k:

$$k^2 c_L^2 A_k^2 + 2k^2 c_T^2 A_k B_k = -\frac{1}{\tau_R^L}A_k - \frac{1}{3}(1 - 2X_k), \tag{159}$$

$$2k^2c_T^2B_k^2 + k^2c_LB_kA_k = -\frac{1}{\tau_R^T}B_k - \frac{1}{3}X_k, \quad (160)$$

$$X_k\left(k^2c_L^2A_k + 2k^2c_T^2B_k\right) = c_T^2k^2B_k + \frac{1}{\lambda}\left[c_L^3 - X_k\left(2c_L^3 + c_T^3\right)\right]. \quad (161)$$

When the energy balance equation (154) is closed with the relations (158), this leads to the dispersion relation for the extended diffusion mode, $\omega_k = \overline{A}_k + 2\overline{B}_k$, where the functions $\overline{A}_k = k^2c_L^2A_k$, and $\overline{B}_k = k^2c_T^2B_k$, satisfy the condition: $\overline{A}_k \to 0$, and $\overline{B}_k \to 0$, as $k \to 0$. The resulting dispersion relation is rather complicated in the general case of the four parameters of the problem, c_L, c_T, τ_R^L and τ_R^T. Therefore, introducing a function $\overline{Y}_k = \overline{A}_k + 2\overline{B}_k$, let us consider the following specific situations of closed equations for the $\overline{Y}_k$ on the basis of the invariance equations (159):

(i). $c_L = c_T = c$, $\tau_R^L = \tau_R^T = \tau_R$ (complete degeneration of the parameters of the L and T subsystems): The system (159) results in two decoupled equations:

$$\overline{Y}_k\left(\tau_R\overline{Y}_k + 1\right) + \frac{1}{3}k^2c^2\tau_R = 0, \quad (162)$$

$$\left(\tau_R\overline{Y}_k + 1\right)^2 + \frac{1}{3}k^2c^2\tau_R^2 = 0. \quad (163)$$

The equation (162) coincides with the eq. (138) for the isotropic case, and its solution defines the coupling of the extended diffusion to a non-hydrodynamic mode. The equation (163) does not have a solution with the required asymptotic $\overline{Y}_k \to 0$ as $k \to 0$, and is therefore irrelevant to the features of the diffusion mode in this completely degenerated case. It describes the two further propagating and damped non-hydrodynamic modes of the Eqs. (154). The nature of these modes, as well of the mode which couples to the diffusion mode well be seen below.

(ii). $c_L = c_T = c$, $\tau_R^L \neq \tau_R^T$ (nondegenerate characteristic time of resistive processes in the L and the T subsystems):

$$\left[\overline{Y}_k\left(\tau_R^L\overline{Y}_k + 1\right) + \frac{1}{3}k^2c^2\tau_R^L\right] \times \left[\left(\tau_R'\overline{Y}_k + 3\right)\left(\tau_R^T\overline{Y}_k + 1\right) + \frac{1}{3}k^2c^2\tau_R^T\tau_R'\right] + \frac{2}{3}k^2c^2\left(\tau_R^T - \tau_R^L\right) = 0, \quad (164)$$

where $\tau_R' = 2\tau_R^L + \tau_R^T$. As $\tau_R^T - \tau_R^L \to 0$, the Eq. (164) tends to the degenerated case (162). At $k = 0$, $\tau_R^L \neq \tau_R^L$, there are four solutions to the Eq. (164). The $\overline{Y}_0 = 0$ is the hydrodynamic solution indicating the beginning of the diffusion mode. The two non-hydrodynamic solutions, $\overline{Y}_0 = -1/\tau_R^L$, and $\overline{Y}_0 = -1/\tau_R^T$, $\overline{Y}_0 = -3/\tau_R'$, are associated with the longitudinal and the

transverse phonons, respectively. The difference in relaxational times makes the latter transverse root nondegenerate, instead there appears the third non-hydrodynamic mode, $\overline{Y}_0 = -3/\tau_R'$.

(iii). $c_L \neq c_T$, $\tau_R^L = \tau_R^T = \tau_R$ (nondegenerate speed of the L and the T sound).

$$\left[\overline{Y}_k\left(\tau_R\overline{Y}_k+1\right)+\frac{1}{3}k^2c_L^2\tau_R\right]\times\left[\left(\tau_R\overline{Y}_k+1\right)^2+\frac{1}{3}k^2c_T^2\tau_R^2\right]+\frac{2}{3}k^2\tau_R\frac{c_L^3(c_T^2-c_L^2)}{2c_L^3+c_T^3}\left(\tau_R\overline{Y}_k+1\right)=0. \tag{165}$$

As $c_T - c_L \to 0$, the Eq. (165) tends to the degenerated case (162). However, this time the non-hydrodynamic mode associated with the transverse phonons is degenerated at $k = 0$.

Thus, we are able to identify the modes in the equations (162) and (163). The non-hydrodynamic mode which couples to the extended diffusion mode is associated with the longitudinal phonons, and is the case (162). The case (163) is due to the transverse phonons. In the nondegenerate cases, (164) and (165), the both pairs of modes become propagating after certain critical values of k, and the behavior of the extended diffusion mode is influenced by all the three non-hydrodynamic modes just mentioned. It should be stressed, however, that second sound mode which is the continuation of the diffusion mode [24], is due to the eq. (162). The results of the above analysis lead to the following discussion:

(i) The examples considered above indicate an interesting mechanism of a *kinetic* formation of the second sound regime from the extended diffusion with the participation of the non-hydrodynamic mode. The onset of the propagating mode shows up as the critical point of the extension of the hydrodynamic solution into the domain of finite k, which was found within the Chapman–Enskog and equivalent approaches. This results concern the situation at the high-temperature edge of the Guyer–Krumhansl window, and are complementary to the coupling between the transversal ballistic mode and second sound at the low-temperature edge [29].

(ii) The crossover from the diffusion-like to the wave-like propagation was previously found in [30] in frames of exact Chapman-Enskog solution to the Boltzmann equation for the Lorentz gas model [7], and for similar models of phonon scattering in anisotropic disordered media [31]. The characteristic common feature of the models studied in [7, 30, 31] and the models [27] is the existence of a gap between the hydrodynamic (diffusive) and the non-hydrodynamic components of the spectrum. Therefore, one can expect that the destruction of the extended diffusion is solely due to

the *existence* of this gap. In applications to the phonon kinetic theory this amounts to the introduction of the relaxation time approximation. In other words, we may expect that the mechanism of crossover from diffusion to second sound in the simple models [27] is identical to what could be found from the phonon–Boltzmann kinetic equation within the relaxation time approximation. However, a remark is in order since the original (i. e., without the relaxation time approximation) phonon kinetic equations are *gapless* (cf., e. g., [26]). On the other hand, most of the works on heat propagation in solids *do* explore the idea of the gap, since it is only possible to speak of the diffusion if such a gap exists. To conclude this point, the following general hypothesis can be expressed: *the existence of the diffusion (and hence of the gap in the relaxational spectrum) leads to its destruction through the coupling to a non-hydrodynamic mode.*

3.7 Nonlinear Grad equations

In the preceding sections, the Chapman-Enskog and other methods were probed explicitly for the linearized Grad equations far beyond the usual Navier–Stokes approximation. This was possible, first of all, because the problem of the reduced description was shaped into a rather simple *algebraic* form. Indeed, the algebraic structure of the stress tensor $\boldsymbol{\sigma}_k(\rho_k, \boldsymbol{u}_k, T_k, \boldsymbol{k})$ and of the heat flux $\boldsymbol{q}_k(\rho_k, \boldsymbol{u}_k, T_k, \boldsymbol{k})$ was fairly simple. However, when we attempt to extend the approach onto the nonlinear Grad equations, the algebraic structure of the problem is no longer simple. Indeed, when we proceed along the lines of the Chapman-Enskog method, for example, the number of *types* of terms, $\nabla\boldsymbol{u}$, $\nabla\nabla\boldsymbol{u}$, $(\nabla\boldsymbol{u})^2$, $\nabla T\nabla\rho$, and so on, in the Chapman-Enskog coefficients $\boldsymbol{\sigma}^{(n)}$ and $\boldsymbol{q}^{(n)}$ demonstrates the combinatorial growth with the order n.

Still, a progress is possible if we impose some rules of selection of relevant terms. As applied to the Chapman-Enskog expansion, these selection rules prescribe to keep only the contributions coming from the terms with a definite structure in each order $\boldsymbol{\sigma}^{(n)}$ and $\boldsymbol{q}^{(n)}$, and to ignore all other terms. This approach can be linked again with the partial summation rules for the perturbation series in many-body theories, where usually terms with a definite structure are summed instead of a series in a whole. Our viewpoint on the problem of extension of the hydrodynamics in the nonlinear case can be expressed as follows: The exact extension seems to be impossible, and, moreover, quite useless because of the lack of a physical transparency. Instead, certain sub-series of the Chapman-Enskog expansion, selected on

clear physical grounds, may lead to less complicated equations, which, at the same time, provide an extension of a certain subclass of hydrodynamic phenomena. This viewpoint is illustrated in this section by considering a sub-series of the Chapman-Enskog expansion which gives the dominating contribution when the flow velocity becomes very large (and thus it is relevant to a high-speed subclass of hydrodynamic phenomena such as strong shock waves).

The approach to the Chapman-Enskog series for the nonlinear Grad equations just mentioned, and which was based on a diagrammatic representation of the Chapman-Enskog method, has been attempted earlier in [14]. In this section, however, we will take the route of the dynamic invariance equations which leads to the same results more directly.

3.7.1 The dynamic viscosity factor

The starting point is the set of one-dimensional nonlinear Grad equations for the hydrodynamic variables ρ, u and T, coupled to the non-hydrodynamic variable σ, where σ is the xx-component of the stress tensor:

$$\begin{aligned}
\partial_t \rho &= -\partial_x(\rho u); && (166)\\
\partial_t u &= -u\partial_x u - \rho^{-1}\partial_x p - \rho^{-1}\partial_x \sigma; && (167)\\
\partial_t T &= -u\partial_x T - (2/3)T\partial_x u - (2/3)\rho^{-1}\sigma\partial_x u; && (168)\\
\partial_t \sigma &= -u\partial_x \sigma - (4/3)p\partial_x u - (7/3)\sigma\partial_x u - \frac{p}{\mu(T)}\sigma. && (169)
\end{aligned}$$

Here $\mu(T)$ is the temperature-dependent viscosity coefficient. We will adopt the form $\mu(T) = \alpha T^\gamma$, which is characteristic to the point-center models of particle's collisions, where γ varies from $\gamma = 1$ (Maxwellian molecules) to $\gamma = 1/2$ (hard spheres), and where α is a dimensional factor.

Even in this model, the Chapman-Enskog expansion appears to be exceedingly complicated in the full setting. Therefore, we address another, more restrictive problem: *What is the leading correction to the Navier–Stokes approximation when the characteristic value of the average velocity is comparable to the thermal velocity?*

Our goal is to compute the correction to the Navier-Stokes closure relation, $\sigma_{\rm NS} = -(4/3)\mu\partial_x u$, for high values of the average velocity. Let us consider first the Burnett correction as from the Eq. (166):

$$\sigma_{\rm B} = -\frac{4}{3}\mu\partial_x u + \frac{8(2-\gamma)}{9}\mu^2 p^{-1}(\partial_x u)^2 - \frac{4}{3}\mu^2 p^{-1}\partial_x(\rho^{-1}\partial_x p). \qquad (170)$$

The correction of the desired type is given by the nonlinear term proportional to $(\partial_x u)^2$. Each further nth term of the Chapman-Enskog expansion contributes, among others, the nonlinear term proportional to $(\partial_x u)^{n+1}$. Such terms can be named *high-speed* terms since they dominate the rest of the contributions in each order of the Chapman-Enskog expansion when the characteristic average velocity is comparable to the heat velocity. Indeed, if U is a characteristic mean velocity, and $u = U\overline{u}$, where $\overline{u}$ is dimensionless, then the term $(\partial_x u)^{n+1}$ receives the factor U^{n+1} which is the highest possible order of U among the terms available in the nth order of the Chapman-Enskog expansion. Simple dimensional analysis leads to a conclusion that such terms are of the form $\mu g^n \partial_x u$, where $g = p^{-1}\mu\partial_x u$ is dimensionless. Therefore, the Chapman-Enskog expansion for the function σ may be formally rewritten as:

$$\sigma = -\mu\left\{\frac{4}{3} - \frac{8(2-\gamma)}{9}g + r_2 g^2 + \ldots + r_n g^n + \ldots\right\}\partial_x u + \ldots \tag{171}$$

The series in the brackets is the collection of the high-speed contributions of interest, coming from *all* the orders of the Chapman-Enskog expansion, while the dots outside the brackets stand for the terms of other nature. Thus, the high-speed correction to the Navier–Stokes closure relation in frames of the Grad equations (166) takes the form:

$$\sigma_{\rm nl} = -\mu R(g)\partial_x u, \tag{172}$$

where $R(g)$ is the function represented by a formal subsequence of Chapman-Enskog terms in the expansion (171). The function R can be viewed also as a dynamic modification of the viscosity μ due to the gradient of the average velocity.

We will now turn to the problem of an explicit derivation of the function R (172). Following the principle of dynamic invariance, we, first, compute the microscopic derivative of the function $\sigma_{\rm nl}$ by substituting the expression (172) into the right hand side of the Eq. (169):

$$\partial_t^{\rm micro}\sigma_{\rm nl} = -u\partial_x\sigma_{\rm nl} - \frac{4}{3}p\partial_x u - \frac{7}{3}\sigma_{\rm nl}\partial_x u - \frac{p}{\mu(T)}\sigma_{\rm nl} = \left\{-\frac{4}{3} + \frac{7}{3}gR + R\right\}p\partial_x u + \ldots, \tag{173}$$

where dots denote the terms irrelevant to the closure relation (172) (such terms appear, of course, because (172) is not the exact closure relation).

Second, computing the macroscopic derivative of the closure relation (172) due to the equations (166), (167), and (168), we have:

$$\partial_t^{\rm macro}\sigma_{\rm nl} = -[\partial_t\mu(T)]R\partial_x u - \mu(T)\frac{dR}{dg}[\partial_t g]\partial_x u - \mu(T)R\partial_x[\partial_t u]. \quad (174)$$

In the latter expression, the time derivatives of the hydrodynamic variables should be replaced with the right hand sides of the Eqs. (166), (167), and (168), where, in turn, the function σ should be replaced by the function $\sigma_{\rm nl}$ (172). After some computation, we come to the following:

$$\partial_t^{\rm macro}\sigma_{\rm nl} = \left\{gR + \frac{2}{3}(1-gR)\times\left(\gamma gR + (\gamma-1)g^2\frac{dR}{dg}\right)\right\}p\partial_x u + \dots \quad (175)$$

Here dots stand again for the terms irrelevant to the present analysis.

Equating the relevant terms in Eqs. (173) and (175), we come to the invariance equation for the function R:

$$(1-\gamma)g^2\,(1-gR)\,\frac{dR}{dg} + \gamma g^2R^2 + \left[\frac{3}{2} + g(2-\gamma)\right]R - 2 = 0. \quad (176)$$

For Maxwell molecules ($\gamma = 1$), Eq. (176) simplifies considerably, and is algebraic:

$$g^2R^2 + \left(\frac{3}{2} + g\right)R - 2 = 0. \quad (177)$$

The solution which recovers the Navier–Stokes closure relation in the limit of small g reads:

$$R_{\rm MM} = \frac{-3 - 2g + 3\sqrt{1 + (4/3)g + 4g^2}}{4g^2}. \quad (178)$$

Function $R_{\rm MM}$ (178) is demonstrated in Fig. 10. Notice that the function $R_{\rm MM}$ is positive for all values of its argument g, as is appropriate for the viscosity factor, while the Burnett approximation to the function $R_{\rm MM}$ violates the positivity.

For other models ($\gamma \neq 1$), the invariance equation (176) is a rather complicated nonlinear ODE with the initial condition $R(0) = 4/3$ (the Navier–Stokes condition). Several ways to derive analytic results are possible. One possibility is to expand the function R into powers of g, in the point $g = 0$. This way will bring us back to the original sub-series of the Chapman-Enskog expansion (see Eq. (171)). Instead, we take the opportunity offered by the

parameter γ. Introduce another parameter $\beta = 1 - \gamma$, and consider the expansion:

$$R(\beta, g) = R_0(g) + \beta R_1(g) + \beta^2 R_2(g) + \ldots .$$

Substituting this expansion into the invariance equation (176), we derive $R_0(g) = R_{\rm MM}(g)$,

$$R_1(g) = -\frac{(1 - gR_0)gR_0 + (3/2)g^2(dR_0/dg)}{2g^2R_0 + g + (3/2)}, \tag{179}$$

etc. That is, the solution for other models is constructed in a form of a series with the exact solution for the Maxwell molecules as the leading term. For hard spheres ($\beta = 1/2$), the result to the first-order term reads: $R_{\rm HS} \approx R_{\rm MM} + (1/2)R_1$. The resulting approximate viscosity factor is given in Fig. 11. The features of the approximation obtained are qualitatively the same as in the case of the Maxwell molecules.

3.7.2 Attraction to the invariant set

Above, we have derived a correction to the Navier–Stokes expression for the stress σ, in the one–dimensional case, for large values of the average velocity u. This correction has the form $\sigma = -\mu R(g)\partial_x u$, where $g \propto \partial_x u$ is the longitudinal rate. The viscosity factor $R(g)$ is a solution to the differential equation (176), subject to a certain initial condition. Uribe and Piña [32] have indicated some interesting features of the invariance equation (176) for the model of hard spheres. In particular, they have found that a numerical integration from the initial point into the domain of negative longitudinal rates is very difficult. What happens to the relevant solution at negative values of g?

Let us denote as $P = (g, R)$ the points of the (g, R) plane. The relevant solution $R(g)$ emerges from the point $P_0 = (0, 4/3)$, and can be uniquely continued to arbitrary values of g, positive and negative. This solution can be constructed, for example, with the Taylor expansion, and which is identical with the relevant sub–series of the Chapman–Enskog expansion. However, the difficulty in constructing this solution numerically for $g < 0$ originates from the fact that the same point P_0 is the point of *essential singularity* of other (irrelevant) solutions to the Eq. (176). Indeed, for $|g| \ll 1$, let us consider $\tilde{R}(g) = R(g) + \Delta$, where $R(g) = (4/3) + (8/9)(\gamma - 2)g$ is the relevant solution for small $|g|$, and $\Delta(g)$ is a deviation. Neglecting in the Eq. (176) all regular terms (of the order g^2), and also neglecting $g\Delta$ in comparison to

Δ, we derive the following equation: $(1-\gamma)g^2(d\Delta/dg) = -(3/2)\Delta$. The solution is $\Delta(g) = \Delta(g_0)\exp[a(g^{-1} - g_0^{-1})]$, where $a = (3/2)(1-\gamma)^{-1}$. The essential singularity at $g = 0$ is apparent from this solution, unless $\Delta(g_0) \neq 0$ (that is, no singularity exists only for the relevant solution, $\tilde{R} = R$). Let $\Delta(g_0) \neq 0$. If $g < 0$, then $\Delta \to 0$, together with all its derivatives, as $g \to 0$. If $g > 0$, the solution blows up, as $g \to 0$.

The complete picture for $\gamma \neq 1$ is as follows: The lines $g = 0$ and $P = (g, g^{-1})$ define the boundaries of the basin of attraction $A = A_- \bigcup A_+$, where $A_- = \{P| -\infty < g < 0, R > g^{-1}\}$, and $A_+ = \{P|\infty > g > 0, R < g^{-1}\}$. The graph $G = (g, R(g))$ of the relevant solution belongs to the closure of A, and goes through the points $P_0 = (0, 4/3)$, $P_- = (-\infty, 0)$, and $P_+ = (\infty, 0)$. These points at the boundaries of A are the points of essential singularity of any other (irrelevant) solution with the initial condition $P \in A$, $P \notin A \bigcap G$. Namely, if $P \in A_+$, $P \notin A_+ \bigcap G$, the solution blows up at P_0, and attracts to P_+. If $P \in A_-$, $P \notin A_- \bigcap G$, the solution blows up at P_-, and attracts to P_0.

The above consideration is supported by a numerical study of the Eq. (176). In Fig. 12, it is demonstrated how the dynamic viscosity factor $R(g)$ emerges as the attractor of various solutions to the invariance equation (176) [the case considered corresponds to hard spheres, $\gamma = 1/2$]. The analytical approximation (179) is also shown in Fig. 12 for the sake of comparison. It provides a reasonable global approximation to the attractor for both positive and negative g. We conclude with a discussion.

(i) The main feature of the above example of extending the hydrodynamic description into a highly non-equilibrium and nonlinear domain can be expressed as follows: this is an *exact partial summation* of the Chapman-Enskog expansion. "Partial" means that the relevant high-speed terms, dominating the other contributions in the limit of the high average velocity, were accounted to all the orders of the original Chapman-Enskog expansion. "Exact" means that, though we have used the formally different route, the result is indeed the sum of the relevant sub-series of the original Chapman-Enskog expansion. In other words, if we now expand the function $R_{\rm MM}(g)$ (178) in powers of g, in the point $g = 0$, we come back to the corresponding series inside the brackets in the Eq. (171). That this is indeed true can be checked up to the few lower orders straightforwardly, though the complete proof requires a more involved analysis, and will be reported elsewhere. As the final comment to this point, we would like to stress a certain similarity between the problem considered above and the frequent situations in many-body problems: there is no a single leading *term* but instead there is the

leading *sub-series* of the perturbation expansions, under certain conditions.

(ii) Let us discuss briefly the features of the resulting hydrodynamics. The hydrodynamic equations are now given by Eqs. (166), (167), and (168), where σ is replaced with $\sigma_{\rm nl}$ (172). First, the correction concerns the nonlinear regime, and thus the linearized form of new equations coincides with the linearized Navier–Stokes equations. Second, the solution (178) for the Maxwellian molecules and the result of approximation (179) for other models suggests that the modified viscosity μR gives a vanishing contribution in the limit of very high values of the average velocity. This feature seems to be of no surprise: if the average velocity is very high in comparison to other characteristic velocities (in our case, to the heat velocity), no mechanisms of momentum transfer are relevant except for the transfer with the stream. However, a cautious remark is in order since the original "kinetic" description are Grad equations (166) and not the Boltzmann equation.

(iii) The invariance equation (176) defines the relevant physical solution to the viscosity factor for all values of g, and demonstrates an interesting phase-space behavior similar to those of finite-dimensional dynamical systems.

4 Conclusions

Up to now, the problem of the exact relationship between kinetics and hydrodynamics remains unsolved. All the methods used to establish this relationship are not rigorous, and involve approximations. In this work, we have considered situations where hydrodynamics is the exact consequence of kinetics, and in that respect, a new class of exactly solvable models of statistical physics has been established.

The main lesson we can learn from the exact solution is the following: The Chapman-Enskog method is the Taylor series expansion approach to solving the appropriate invariance equation. Alternative iteration methods are much more robust for solving this equation. Therefore, it seems quite important to develop approaches to the problem of reduced description based on the principle of dynamic invariance rather than on particular methods of solving the invariance equations. The exact solutions where these questions can be answered in all the quantitative details provide a sound motivation for such developments. Interested reader is directed to the paper [23] where a general formulation of the reduced description from the standpoint of constructing invariant manifolds has been developed, as well as to some

recent applications in the problems of Boltzmann kinetic theory, chemical kinetics, and polymer kinetic theory are given [33, 34, 35, 36].

5 Acknowledgement

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Figure 1: Real-valued root of Eq. (33) as a function of k^2

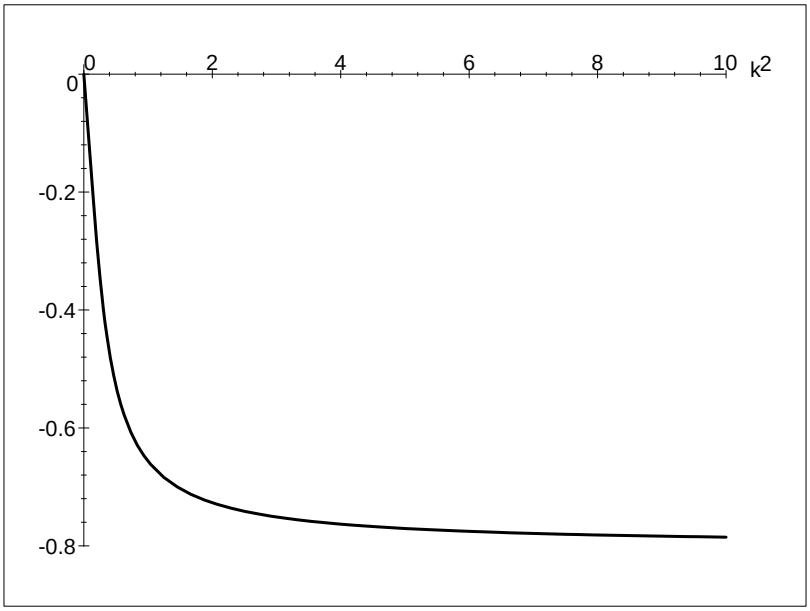

Figure 2: Attenuation rates for the $1D10M$ Grad system. Solid: Exact summation of the Chapman–Enskog Expansion. Dots: The Navier–Stokes approximation. Dash: The super–Burnett approximation. Circles: Hydrodynamic and non-hydrodynamic modes of the $1D10M$ Grad system.

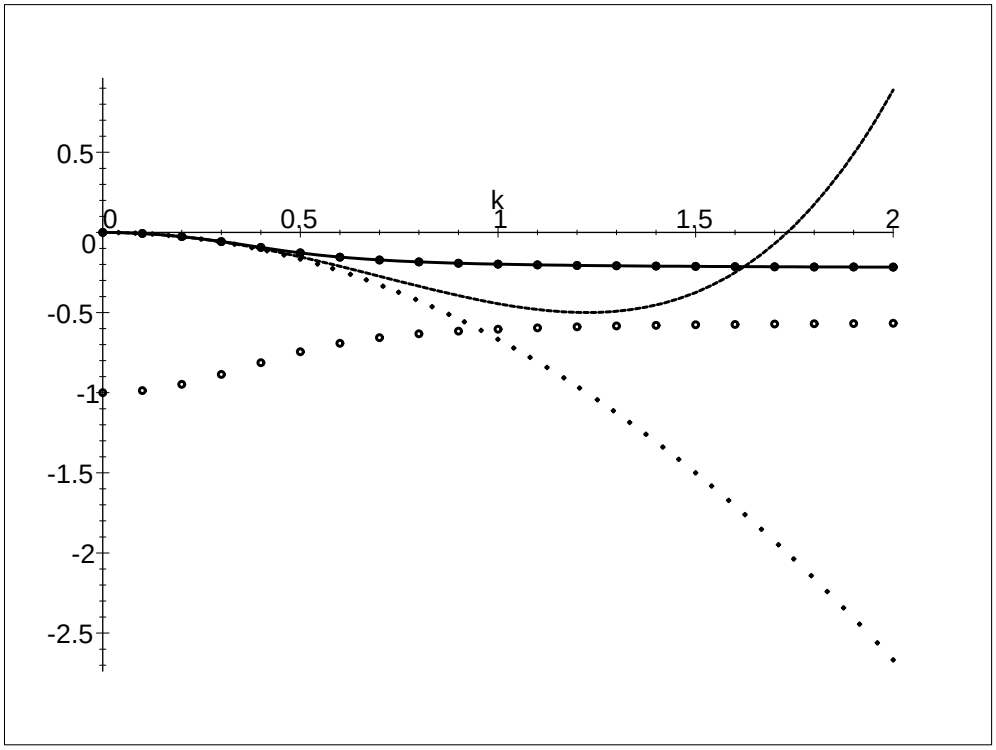

Figure 3: Attenuation rates for the $3D10M$ Grad system. Bold: The acoustic branch, exact summation. Dots: The acoustic branch, Navier–Stokes approximation. Circles: The acoustic branch, super-Burnett approximation. Solid: The diffusion branch, exact summation. Dash: The critical mode of the $3D10M$ Grad system.

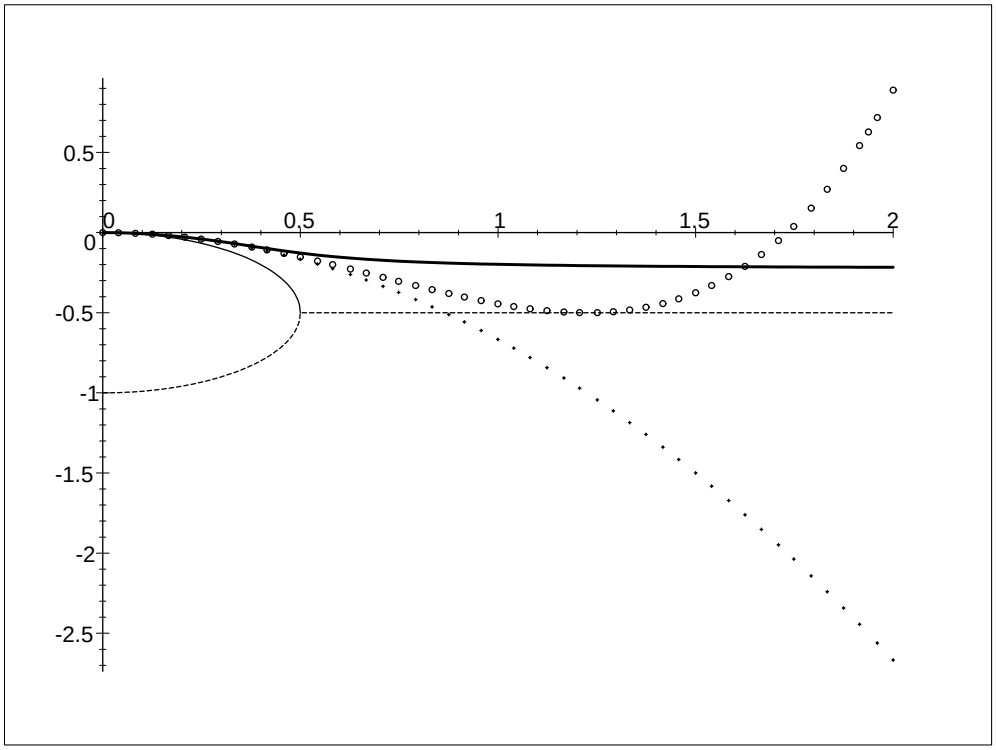

Figure 4: Attenuation rates for the partial summing. Solid: The regularized Burnett approximation. Dash: The regularized super-Burnett approximation. Circles: The super-Burnett approximation. Dots: The exact summation.

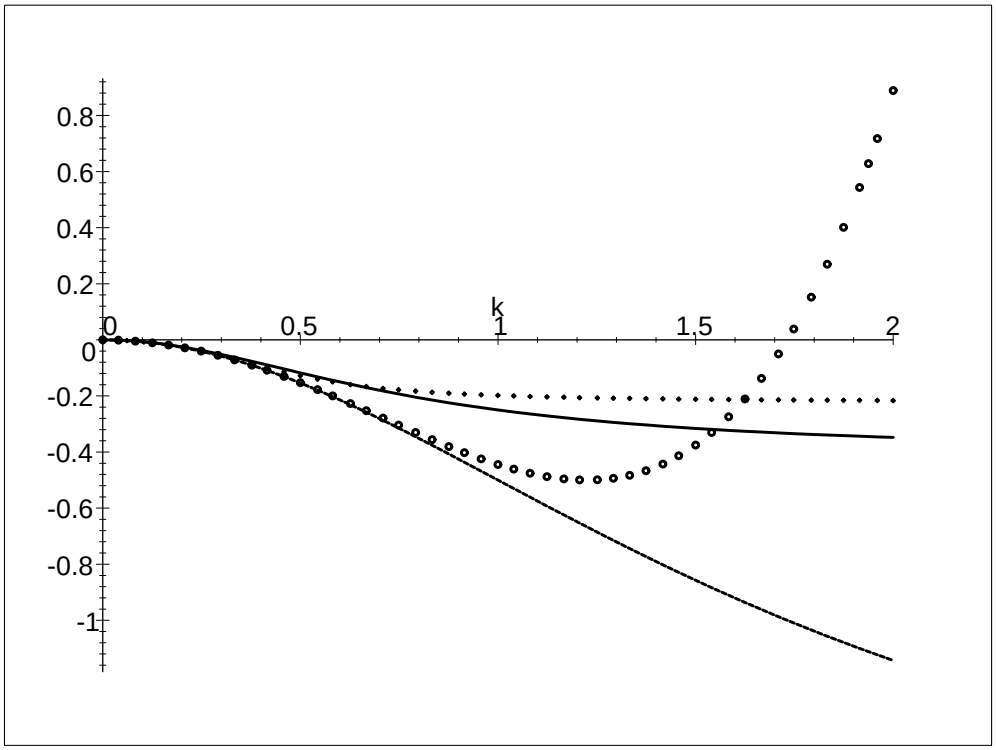

Figure 5: Attenuation rates for the Newton method with the Navier–Stokes approximation as the initial condition. Dots: The Navier–Stokes approximation. Solid: The first and the second iterations of the invariance equation. Circles: The exact solution to the invariance equation. Diamond: The super-Burnett approximation.

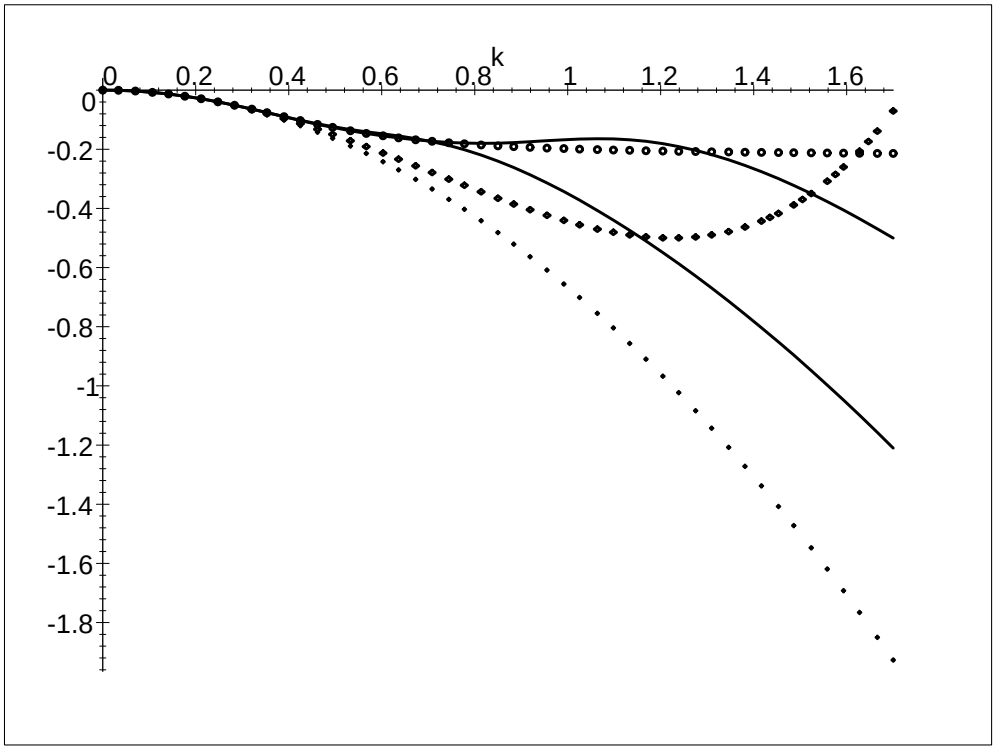

Figure 6: Attenuation rates with the regularized Burnett approximation as the initial condition for the Newton method. Dots: The regularized Burnett approximation, or the first Newton iteration with the Euler initial condition (see text). Solid: The first and the second Newton iterations with the regularized Burnett approximation as the initial condition. Circles: The exact solution to the invariance equation.

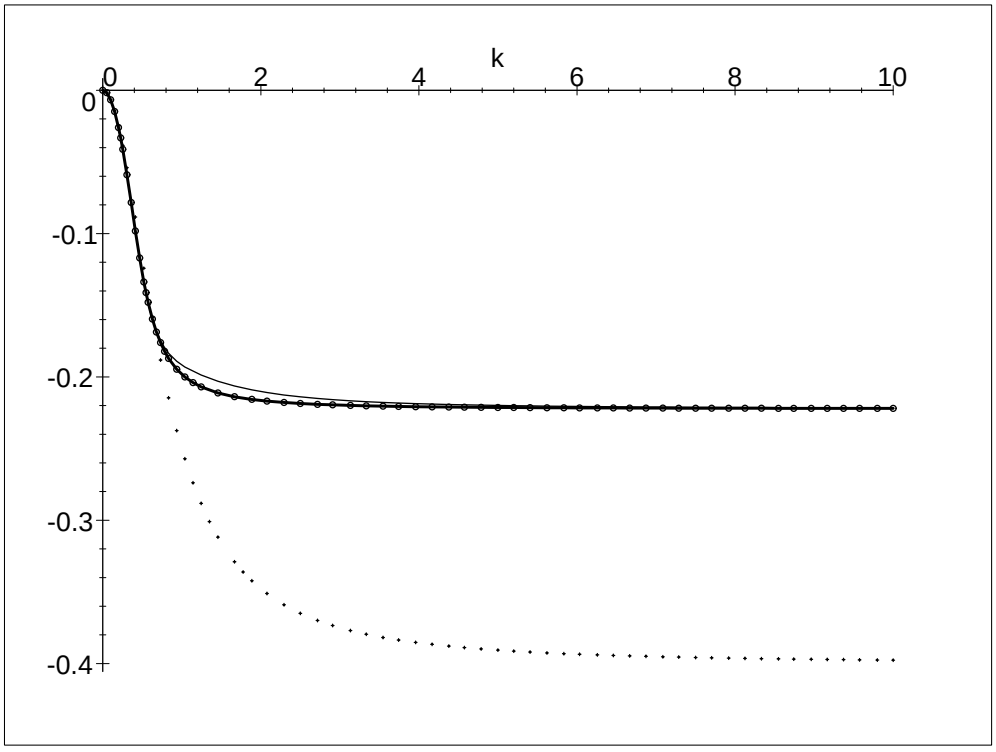

Figure 7: Attenuation rates with the regularized super-Burnett approximation as the initial condition for the Newton method. Dots: The regularized super-Burnett approximation. Solid: The first and the second Newton iterations. Circles: The exact solution to the invariance equation.

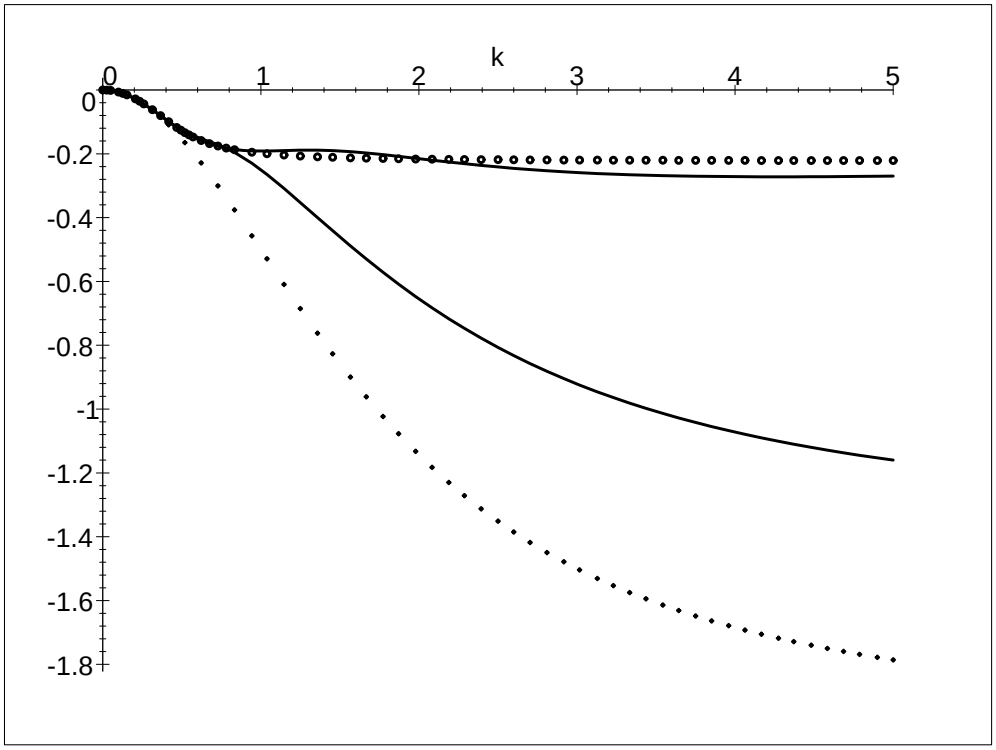

Figure 8: The diffusion mode with the Euler initial approximation for the invariance equation. Dot-dash: The the first iteration. Dots: The second iteration. Circles: The third iteration. Solid: The exact solution. Dash: The critical mode.

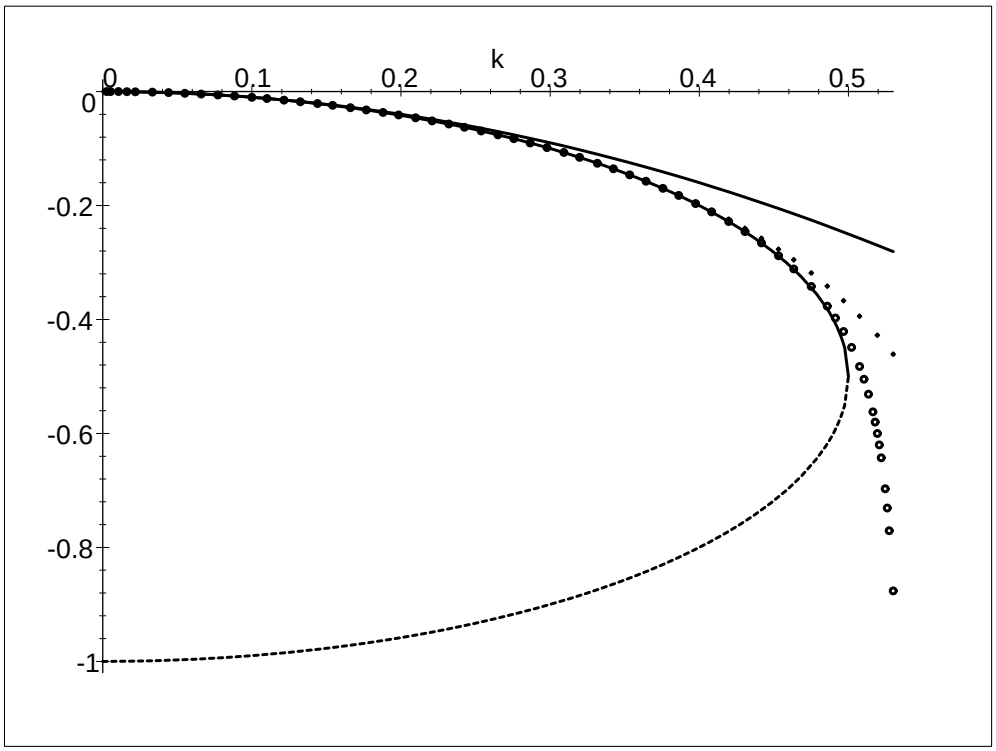

Figure 9: Solutions to the Eq. (123). Circles: Numerical solutions. Dots: The first Newton iteration. Solid: The 4th Newton iteration.

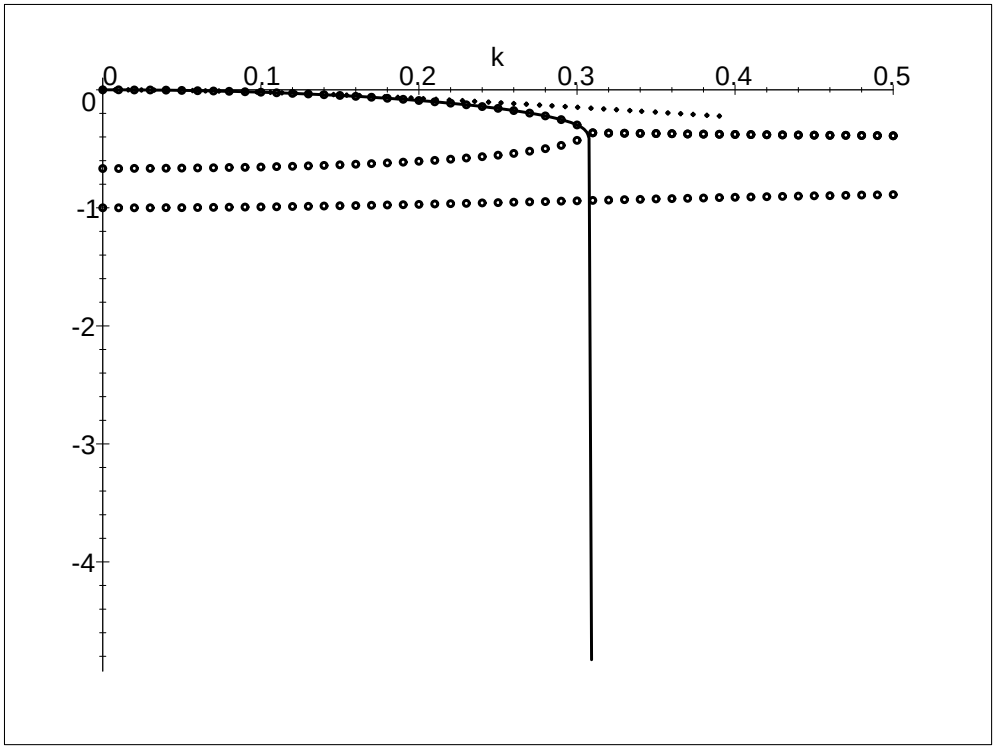

Figure 10: Viscosity factor $R(g)$ for Maxwell molecules. Solid: exact solution. Dash: the Burnett approximation. Dots: the Navier-Stokes approximation.

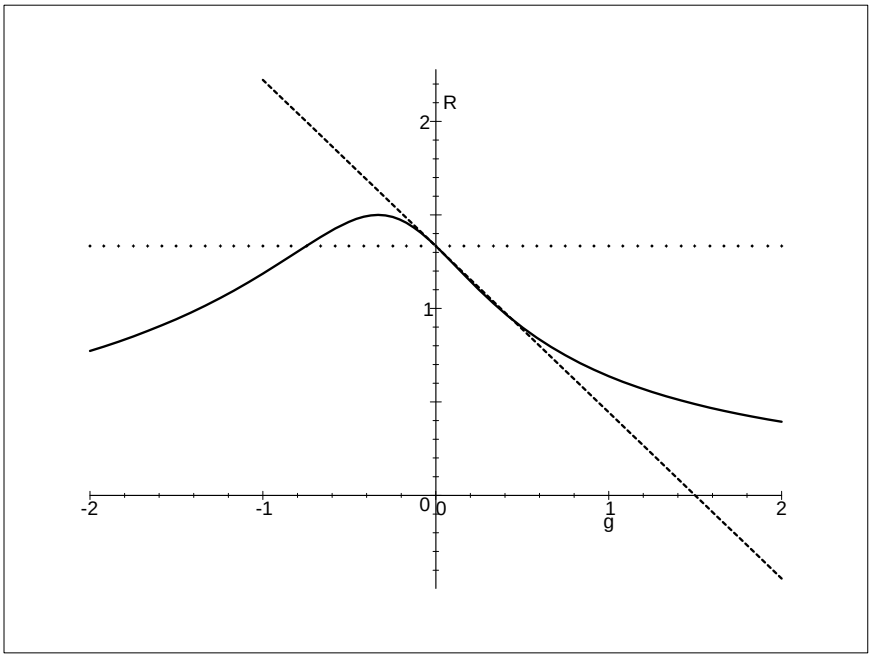

Figure 11: Viscosity factor $R(g)$ for hard spheres. Solid: the first approximation. Dash: exact solution for Maxwell molecules.

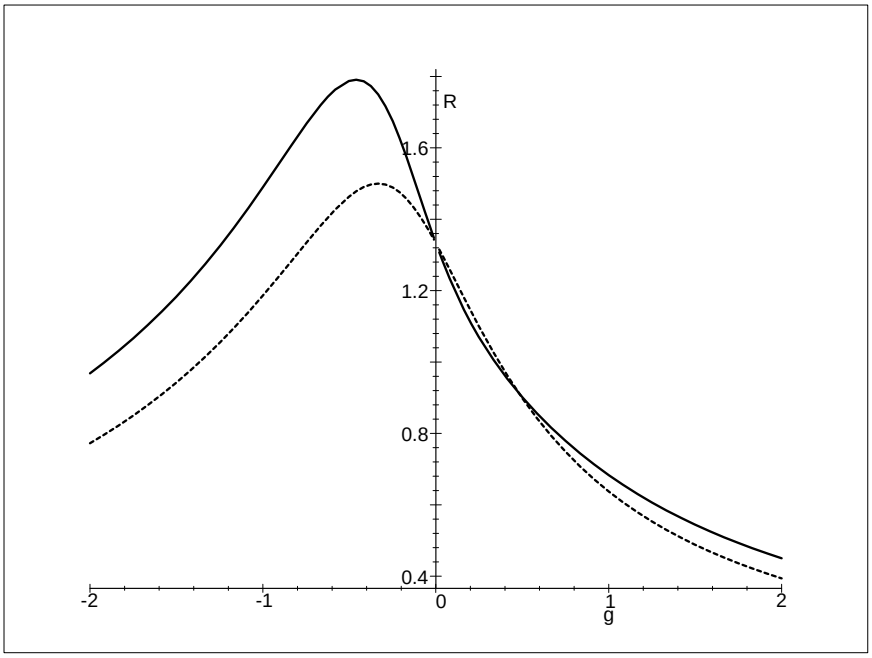

Figure 12: Viscosity factor as an attractor. Solid lines: numerical integration with various initial points (crosses). Two poorly resolved lines correspond to the initial conditions $(-100, 0)$ and $(-100, 3)$. Circles: Taylor expansion to the 5th order. Dots: the analytical approximation of (179). Dash: boundaries of the basin of attraction.