The Finite Element Method for Convection-Dominated Convection-Diffusion Problems

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Introduction

The work is devoted to numerical methods for solving singularly perturbed problems for the convection-diffusion equation with the highest derivatives multiplied by a small parameter. In this case the order of the non-perturbed (singular) equation is one less than of the original (perturbed) equation. Therefore the boundary conditions of the perturbed problem are not all fulfilled for the singular one. Some of these conditions are superfluous that leads to the fast variation of the solution in a small vicinity of corresponding parts of a boundary. As a result, the standard finite difference and finite element methods on a uniform grid either are unstable or give poor accuracy for a small parameter of diffusion.

Some data on the asymptotic analysis of the influence of a small parameters in differential equations go back to L.Euler. The modern theoretical and practical investigations have their origin in A.N.Tikhonov's works of 1940s ([48], [49], [50]). The systematic development of methods for solving singularly perturbed problems started in the late 1960s.

In studies of the properties of a differential problem, the methods of the asymptotic expansion with respect to a small parameter were applied (see [14], [33], [16], [42], [17], [37], [41], [43], [87], [18] and the reviews in them) such as the method of the inner and outer expansions ([14] – 1967), the method of M.I.Vishik and L.A.Lusternick ([19] – 1952 and also [44], [19], [52]), and the method of boundary functions being the generalization of the latter one ([15] and [16] - 1960s, and also [11], [17], [13], [18]).

The use of the standard finite difference and finite element methods for solving singularly perturbed problems failed because of poor accuracy and instability of the discrete analogues. Detailed investigations in this field can be found in [109], [73], [101], [62], [23], [4], as well as in the monograph [117] where the present state of numerical methods for solving singularly perturbed problems is covered in considerable detail.

For the problems considered here the constants in the estimates of the convergence of the classical methods, as a rule, depend on a small parameter and increase indefinitely when the parameter approaches zero [4]. Therefore, these methods can not be applied as mentioned above.

There are several approaches to overcome these difficulties. By convention they can be divided into two groups. The first group is made up of various fitted methods in which the coefficients of a difference scheme in the finite difference method or the parameters of a bilinear form and basis functions in the finite element method are chosen with the use of a-priori information on the behavior of the solution of a differential problem (see, e.g., [23]). The second group consists of standard methods on non-uniform grids which are a-priori given or a-posteriori adapted in the process of numerical integration (see, e.g., [5], [57], [36]).

The first attempts to achieve higher-order accuracy are connected with the use of the upwind scheme. The basic idea of this method is to apply an appropriate approximation of the convective term (by the directed differences) and to add artificial viscosity along the streamline direction. It has been proved that this approach leads to the second order convergence for moderate values of the diffusion parameter and to the convergence of only the first order when the value of the parameter is comparable with or less than a mesh size ([102], [127], [124], [58], [69]).

The construction of the methods uniformly convergent with respect to a small parameter is of great importance in numerically solving the problems with a boundary layer. The exponentially fitted methods satisfy this property. They are constructed using the information on a form of the boundary layer component of a solution ([25], [26], [24], [59], [78], [79], [80]). Another way to construct uniformly convergent difference schemes is to use the analytical solution of an equation with constant coefficients. This approach proposed by D.N.Allen and R.V.Southwell [59] is based on the proximity of the original problem to the approximating one with piecewise constant coefficients and gives a discrete problem similar to the exponentially fitted scheme of A.M.Il'in [25]. In the context of this approach, mention should be made of the method of integral identities with special weight functions [38]. This method is constructed in much the same way as the truncated difference schemes of A.A.Samarskii [45].

One more way to achieve higher-order accuracy of the finite difference method outside a boundary layer is connected with increasing the number of nodes in a stencil ([22], [82], [105]). This complicates the stability analysis as well as the two- and three-dimensional generalizations.

As we noted above, the alternative way to construct uniformly convergent methods is to use special grids. First of all, these are the grids proposed by N.S.Bakhvalov [5]. They are logarithmically refined inside the boundary layer. The construction of these grids is based on the estimates of the derivatives of a solution or on the fact that the difference of the values of a solution at any two neighboring nodes of a grid is uniformly bounded with respect to the parameter ([35], [36]). As a rule, this way leads to a nonlinear algebraic equation for some parameters of this function. Therefore various explicit approximations of logarithmic function are used to construct the Bakhvalov grids ([129], [130], [131], [6], [7], [85], [86]).

In [54] and [123] G.I.Shishkin proved that for the problems with a parabolic boundary layer it is impossible to construct a fitted difference scheme with a compact stencil that converges uniformly with respect to a small parameter. Besides, in [54] the nonuniform grid with a piecewise constant mesh size decreasing in a boundary layer was proposed. In this case the upwind scheme is convergent with order $N^{-1}\ln N$ where N is the number of nodes of the grid. For singularly perturbed problems, the general concept of the proof of the uniform convergence of the classical difference schemes on these grids is presented in the monograph [57] by G.I.Shishkin. In [55], [56], [89], [90], [91], [81] this approach is applied to a wide range of singularly perturbed problems in the finite difference framework and in [113], [119], [125], [111] the Shishkin grids are discussed in the context of finite element method.

All these approaches applied to the finite element method together with the specific finite element techniques give a number of tools for numerical solving singularly perturbed problems.

The upwind scheme in the finite element method has several modifications. For example, in the Petrov-Galerkin method [62] the standard piecewise linear trial functions but the piecewise quadratic test functions are used ([74], [92], [93], [94]). K.Morton proposed to construct test functions which yield a simmetric (or nearly simmetric) discrete problem because in this case the Ritz-Galerkin technique is optimal with respect to the energy norm [101]. For one-dimensional problems this method works well but it is difficult to generalize it to higher-dimensional problems ([108], [109], [110]). Mention should be made of the method proposed by M.Tabata in [126] where the convective term is approximated on the upwind elements only ([71], [60], [61]).

T.Hughes and A.Brooks proposed the method using additional viscosity in the streamline direction ([72], [95]). Instead of the standard bilinear form in the Petrov-Galerkin method they considered some its approximation with an additional term introducing additional viscosity in the streamline direction. As a result, the pointwise convergence of the second order can be achieved on the grids oriented in the streamline direction ([107], [98], [99], [100], [60], [132], [133], [134]). This approach is equivalent to the use of the Galerkin method on the special space being the orthogonal product of the space of piecewise linear functions and that of "bubble functions" [70].

We also mention the method of the additive selection of boundary layer functions ([3], [1], [2]). The basic idea of this method is to add one or two exponential functions with a non-local support, that provides a successful approximation of the boundary layer component, to the standard piecewise linear basic.

In the context of adding artificial viscosity, the least squares method can be applied ([96], [97], [83], [84]). A drawback of this method is that when using piecewise polynomial elements, the assumption that the trial and test functions belong to Sobolev's space $W_2^2(\Omega)$ requires the use of finite elements of $C^1(\Omega)$; but the construction of these elements on an arbitrary triangulation is not easy. Besides, the number of nonzero entries of the stiffness matrix increases.

The application of exponential fitting to the finite element method is represented by two different approaches. In the first approach special piecewise exponential functions are used ([112], [115], [116]). They approximate the smooth component of a solution somewhat worse than piecewise linear ones but give a considerably better approximation of the boundary layer component. This enables to achieve higher-order accuracy in the Galerkin method. We also mention the non-conforming finite element method [118] where discontinuous exponential finite elements are used.

Another approach that extends difference exponential fitting was proposed for the one-dimensional convection-diffusion equation in [121]. The further development of this method is the subject of this work. The basic idea of this approach is to use the standard piecewise linear finite elements on a uniform grid, applying special fitted quadrature rules to approximate the boundary layer component. As a result, the approximate solution converges to the piecewise linear interpolant of the exact one both in the mean square and in the uniform norms.

Recently in the finite difference and finite element methods, adaptive grids are used. They are constructed using a-posteriori information on the approximate solution obtained on a uniform or coarse grid. To estimate the quality of a numerical solution, special functionals named estimators are applied. A number of estimators is proposed in the literature ([63], [64], [67], [65], [75], [66], [76], [77], [128], [88]).

The present work is devoted to the construction and justification of exponentially fitted schemes in the finite element method for the Dirichlet problem for the convection-dominated convection-diffusion equation. Now we outline the basic idea of this approach.

Let Ω be a one- or two-dimensional domain with a piecewise smooth boundary Γ. We consider the Dirichlet problem

$$Lu \equiv -\varepsilon\,\Delta u + \frac{\partial}{\partial x}(b(x)u) = f \qquad \text{in} \quad \Omega, \tag{1}$$

$$u = 0 \qquad \text{on} \quad \Gamma \tag{2}$$

where $\varepsilon \ll 1$ is a positive parameter. The weak formulation of (1) – (2) is given as follows: *find* $u \in H_0^1(\Omega)$ *such that*

$$a(u,v) = (f,v) \qquad \forall\, v \in H_0^1(\Omega). \tag{3}$$

Here $a(\cdot,\cdot)\colon\, H_0^1(\Omega) \times H_0^1(\Omega) \to R$ is the bilinear form determined by

$$a(u,v) = \int_\Omega \left(\varepsilon\,\nabla u \nabla v - bu\frac{\partial v}{\partial x}\right)\, d\Omega$$

and $(\cdot,\cdot)$ is the inner product in $L_2(\Omega)$. We represent the solution of (1)–(2) as

$$u = v + \rho \tag{4}$$

where v is the smooth component of the solution which provides a good approximation of u outside the boundary layer and ρ is the boundary layer component which varies fast in a narrow region near some parts of the boundary.

We choose a finite-dimensional space of test functions $T_h \in H_0^1(\Omega)$ with the basis $\{\varphi_j\}_{j=1}^M$. We consider the discrete problem corresponding to (3): *find* $u \in T_h$ *such that*

$$a^h(u^h,v^h) = f^h(v^h) \qquad \forall\, v^h \in T_h. \tag{5}$$

Here $a^h(\cdot,\cdot) : T_h \times T_h \to R$ is a bilinear form approximating $a(\cdot,\cdot)$ and $f^h : T_h \to R$ is the approximation of the inner product $(f,\cdot)$. In the usual investigation of (5), the following expansion of the error is used:

$$\begin{aligned} a^h(u^h - u^I, w^h) &= a^h(u^h,w^h) - a^h(u^I,w^h) + a(u^I,w^h) \\ &\quad - a(u^I,w^h) + a(u,w^h) - a(u,w^h) \\ &= f^h(w^h) - f(w^h) + (a - a^h)(u^I,w^h) + a(u - u^I, w^h). \end{aligned} \tag{6}$$

Here u^I is the interpolant of the solution in T_h. In this case, the estimate of the last term in (6) increases indefinitely as ε decreases because the solution contains the boundary layer component ρ. The main point of the presented approach ([121]) is to construct the special approximation of a^h in order to reduce the error $a(\rho, w^h) - a^h(\rho^I, w^h)$ in the estimate

$$\begin{aligned} a^h(u^h - u^I, w^h) &= \left(f^h(w^h) - f(w^h)\right) + \left(a(u, w^h) - a^h(u^I, w^h)\right) \\ &= \left(f^h(w^h) - f(w^h) + a(v, w^h) - a^h(v^I, w^h)\right) \\ &+ \left(a(\rho, w^h) - a^h(\rho^I, w^h)\right). \end{aligned}$$

The further development of this approach is as follows. Firstly, for the approximation of the boundary layer component we apply the quadrature rules of higher accuracy. Secondly, we use the special approximation of the right-hand side to eliminate the main term of the error of the quadrature rule on the smooth component.

In the first chapter this approach is applied to the one-dime-sional convection-diffusion equation with the highest derivative multiplied by a small parameter. First we construct the discrete problem based on the linear quadrature rule for the approximation of the convection term and use the special quadrature rule for the approximation of the right-hand side. Next we apply the nonlinear quadrature rule. For the obtained grid problems the second order convergence in the uniform norm is proved for small values of ε.

The extension to the two-dimensional case in **the second chapter** complicates the behavior of a solution. Along with a regular boundary layer which is locally described by an ordinary differential equation, a parabolic boundary layer can arise near some parts of the boundary. It satisfies a parabolic differential equation.

In Section 2.1 the general characteristic of the differential problem is given. The comparison principle is proved for the family of differential equations with the boundary conditions of two types. The weak formulation of the problem is presented. **In Section 2.2** the problem free of a parabolic boundary layer of order 0 is considered. Some estimates of the solution and its derivatives are obtained by the comparison principle. On a uniform grid the discrete problem based on the Galerkin method with piecewise linear elements is constructed using the fitted quadrature rules. The first order of convergence is proved.

In Section 2.3 we investigate the problem with regular and parabolic boundary layers. In this case fitting methods fail ([54]). Therefore, together with the fitted quadrature rules for the approximation of the regular boundary layer, we use a special grid refined in the parabolic boundary layer. This grid is similar to that of Bakhvalov type but in the construction of the grid the generating function is not used. Moreover, the distribution of nodes is given by the one-parameter recurrent formula. The stability and convergence results for this problem are obtained on this grid. In this case the first order convergence is also proved.

In the third chapter the numerical results are discussed.

Section 3.1 is devoted to the numerical experiments in the one-dimensional case. The results demonstrate high accuracy and the advantage of the proposed method over well-known ones. Further, some modifications of the Gauss-Seidel method for solving the two-dimensional discrete problem are considered. The calculations were carried out on the grids of three types. In the two-dimensional problem the exact solution was presented in the form of infinite series. All numerical results on stability and convergence are in close agreement with the theoretical ones.

1 One-dimensional convection–diffusion problem

In this chapter the boundary value problem for the ordinary differential convection-dominated convection-diffusion equation is considered. In spite of its simplicity, this problem has the characteristic feature of the convection-dominated problems, namely, a boundary layer. As a result, most of the classical finite difference and finite element methods fail. Thus, we have a simple object to demonstrate in detail all characteristic properties of the problem as well as of the numerical methods proposed.

1.1 The differential problem and its properties

1.1.1 Boundary Layer

Consider the ordinary differential equation with the highest derivative multiplied by a small parameter

$$Lu \equiv -\varepsilon u'' + (b(x)u)' = f(x) \quad \text{on} \quad (0,1), \tag{1.1}$$

$$0 < B_0 \le b(x) \le B_1 \quad \text{on} \quad [0,1] \tag{1.2}$$

satisfying the Dirichlet boundary condition

$$u(0) = u_0, \quad u(1) = u_1. \tag{1.3}$$

The functions b and f are assumed to be sufficiently smooth

$$b \in C^2[0,1], \quad f \in C^2(0,1). \tag{1.4}$$

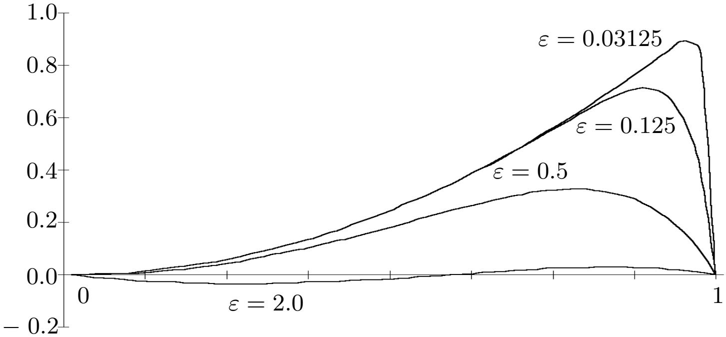

Fig. 1: The appearance of a boundary layer with $\varepsilon \to 0$.

The small coefficient $0 < \varepsilon \ll 1$ of the diffusion term causes the derivatives of the solution to increase exponentially at $x = 1$ ([19], [23]). The appearance of a boundary layer is illustrated in Fig. 1. Here the exact solutions of the problem

$$-\varepsilon u'' + ((1+2x)u)' = 6x^2 + 2x - 2\varepsilon + 2\frac{\exp(-2/\varepsilon)}{1-\exp(-2/\varepsilon)}, \quad x \in (0,1),$$
$$u(0) = u(1) = 0,$$

are shown for four different values of the diffusion parameter ε.

1.1.2 The asymptotic expansion of the solution

There are many techniques to describe the asymptotic behavior of the solution of the problem (1.1)-(1.3) for small ε. We use the method of expansion in powers of ε proposed by M.I.Vishik and L.A.Lusternik. We introduce the new ('fast') variable $\tau = \dfrac{1-x}{\varepsilon}$ to describe the of boundary layer effects near $x = 1$.

Applying the Vishik - Lusternik technique, we obtain the following expansion of the solution

$$u(x) = v_0(x) + \tilde{\rho}_0(\tau) + \varepsilon\left(v_1(x) + \tilde{\rho}_1(\tau)\right) + \varepsilon^2 \tilde{z}(x)$$

where v_0 and εv_1 are smooth components which give a good approximation of the solution outside the boundary layer, $\tilde{\rho}_0$ and $\varepsilon\tilde{\rho}_1$ are boundary layer terms, and $\varepsilon^2\tilde{z}(x)$ is a remainder term. Here, $v_0(x)$ is the solution of the reduced problem

$$(bv_0)' = f \quad \text{on } (0,1), \quad v_0(0) = u_0 \tag{1.5}$$

and $v_1(x)$ is the solution of the problem

$$(bv_1)' = {v_0}'' \quad \text{on } (0,1), \quad v_1(0) = 0. \tag{1.6}$$

The boundary layer functions are described by means of the problems

$$-\tilde{\rho}_0''(\tau) + b(1)\tilde{\rho}_0'(\tau) = 0, \quad \tilde{\rho}_0(0) = u_1 - v_0(1), \quad \lim_{\tau\to\infty} \tilde{\rho}_0(\tau) = 0,$$

and

$$-\tilde{\rho}_1''(\tau)+b(1)\tilde{\rho}_1'(\tau) = b'(1)\tilde{\rho}_0(\tau)-\tau b'(1)\tilde{\rho}_0'(\tau), \quad \tilde{\rho}_1(0) = -v_1(1), \lim_{\tau\to\infty} \tilde{\rho}_1(\tau) = 0$$

with the solutions

$$\tilde{\rho}_0(\tau) = (u_1 - v_0(1)) \exp(-b(1)\tau), \tag{1.7}$$
$$\tilde{\rho}_1(\tau) = \big((u_1 - v_0(1))b'(1)\tau^2/2 - v_1(1)\big) \exp(-b(1)\tau). \tag{1.8}$$

The functions $\tilde{\rho}_k(\tau)$ are defined for $\tau \geq 0$ but for small values of ε they differ from zero only in a small vicinity of the point $\tau = 0$. Therefore we multiply $\tilde{\rho}_0(\tau)$ and $\tilde{\rho}_1(\tau)$ by the cut-off function from $C^2[0,1]$ defined as

$$s(t) = \begin{cases} 0, & t \leq 1/3, \\ \text{monotonically increases on } [1/3, 2/3], & \\ 1, & t > 2/3 \end{cases} \tag{1.9}$$

and pass to the variable x:

$$\rho_0(x) = s(x)\tilde{\rho}_0(\tau), \quad \rho_1(x) = s(x)\tilde{\rho}_1(\tau). \tag{1.10}$$

As a result, we get the following expansion of the solution of (1.1)–(1.3) for small ε

$$u(x) = v_0(x) + \rho_0(x) + \varepsilon\,(v_1(x) + \rho_1(x)) + \varepsilon^2 z(x). \tag{1.11}$$

1.1.3 The estimates of the remainder term

We introduce the following norms for the function defined on the segment $[0,1]$

$$||v||_p = \begin{cases} \left(\int\limits_0^1 |v|^p\,dx\right)^{1/p}, & 1 \leq p < \infty, \\ \sup\limits_{[0,1]} \mathrm{vrai}|v|, & p = \infty. \end{cases} \tag{1.12}$$

The following theorem gives the estimate of the remainder term $z(x)$ in the uniform norm.

Theorem 1. *Under the conditions* (1.2), (1.4) *the remainder term* $z(x)$ *of the expansion* (1.11) *obeys the estimate*

$$||z||_\infty \leq c_1 \quad {}^{*)} \tag{1.13}$$

with a constant c_1 *independent of* ε.

Proof. We express z from (1.11):

$$z(x) = \frac{1}{\varepsilon^2}\left(u(x) - v_0(x) - \rho_0(x) - \varepsilon\left(v_1(x) + \rho_1(x)\right)\right).$$

We substitute this expression in (1.1) and use the expansion of the functions $b(x)$ and $b'(x)$ into the Taylor series at 1. Collecting similar terms, we get

$$Lz(x) = \tilde{f} \equiv a_0 + a_1\frac{1}{\varepsilon}A + a_2\frac{1-x}{\varepsilon}A + a_3\frac{(1-x)^2}{\varepsilon^2}A \tag{1.14}$$

where $a_0(x)$, $a_1(x)$, $a_2(x)$, and $a_3(x)$ are some bounded functions and $A(x) = \exp(-(1-x)b(1)/\varepsilon)$. Since the functions $t\exp(-t)$ and $t^2\exp(-t)$ are bounded on $[0,1]$ by some constants, the last two terms in $\tilde{f}$ are also bounded.

The calculation of $z(0)$ and $z(1)$ by means of boundary conditions for the boundary layer components ρ_0 and ρ_1 and the use of properties of the cut-off function $s(t)$ yield:

$$z(0) = z(1) = 0. \tag{1.15}$$

The problem (1.14)-(1.15) satisfies the comparison principle [121]. Take

$$y(x) = \exp(\sigma x)\left(\gamma_0 + \gamma_1 x + \gamma_2\exp\left(-\frac{(1-x)B_1}{2\varepsilon}\right) + \gamma_3\exp\left(-\frac{(1-x)B_1}{4\varepsilon}\right)\right)$$

as a barrier function where

$$\sigma = 1 + \max_{x\in[0,1]}\frac{|b'| - b'}{2b}.$$

Then

$$Ly(x) \geq |Lz(x)| \quad \text{on} \quad (0,1), \qquad y(0) \geq 0, \quad y(1) \geq 0.$$

Hence by the comparison principle we have

$$|z(x)| \leq y(x) \leq \max_{x\in[0,1]} y(x) = c_1.$$

---

*) In what follows, c_i denote constants which are independent of ε, x, and of h at a later time.

This completes the proof. $\square$

Along with the expansion (1.11) consider the asymptotic expansion

$$u(x) = v_0(x) + \rho(x) + \varepsilon z_1(x) \tag{1.16}$$

which will be used to derive the quadrature rule in Section 3. Here $v_0(x)$ is the solution of the reduced problem as before. The boundary layer component is taken like in [121] in the form

$$\rho(x) = s(x)\,(u_1 - v_0(1))\exp(-(1-x)b(x)/\varepsilon). \tag{1.17}$$

For the remainder term $z_1(x)$ the following estimate is proved in [121].

Theorem 2. *Assume that the conditions* (1.2), (1.4) *hold and* z_1 *is given by* (1.16) – (1.17)*. Then there is a positive constant* c_4 *such that the estimate*

$$|z_1^{(j)}| \le c_4 \quad on \quad [0,1], \quad j = 0, 1, \tag{1.18}$$

holds for sufficiently small ε*.*

We also evaluate the difference between the functions ρ_0 and ρ.

Lemma 3. *Let* ρ_0 *and* ρ *be the boundary layer components of order* 0 *given by the formulae* (1.10) *and* (1.17) *respectively. Then there is a positive constant* c_5 *such that the estimate*

$$|\rho_0 - \rho| \le c_5\varepsilon \quad on \quad [0,1] \tag{1.19}$$

holds for sufficiently small ε*.*

Proof. By the mean-value theorem, the following inequality holds for any $x \in [0,1]$:

$$\begin{aligned} &|\exp(-(1-x)b(1)/\varepsilon) - \exp(-(1-x)b(x)/\varepsilon)| \\ &\quad \le (b(1) - b(x))\frac{1-x}{\varepsilon}\exp(-(1-x)\tilde{b}/\varepsilon) \end{aligned}$$

where $\tilde{b} \in [B_0, B_1]$. Since

$$|b(1) - b(x)| \le |1-x|\|b'\|_\infty \le c_6|1-x|$$

and $t^2\exp(-\alpha t) \le c_7$ for all $\alpha \ge 0$ and $t \in [0,\infty)$, the following inequality holds:

$$\begin{aligned} &|b(1) - b(x)|\frac{1-x}{\varepsilon}\exp(-(1-x)\tilde{b}/\varepsilon) \\ &\le c_6\varepsilon\left(\frac{1-x}{\varepsilon}\right)^2\exp(-(1-x)\tilde{b}/\varepsilon) \le c_6c_7\varepsilon. \end{aligned}$$

This completes the proof. $\square$

1.1.4 The weak formulation. The Petrov-Galerkin method

Multiply (1.1) by an arbitrary function $v \in H_0^1(0,1)$. By applying Green's formula, we obtain the weak formulation: *find* $u \in H^1(0,1)$ *which satisfies the boundary condition* (1.3) *and the equality*

$$a(u,v) = (f,v) \quad \forall\, v \in H_0^1(0,1). \tag{1.20}$$

Here $a(\cdot,\cdot)$: $H^1(0,1) \times H_0^1(0,1) \to R$ is the bilinear form

$$a(u,v) = \int_0^1 (\varepsilon u' - bu)\, v'\, dx, \tag{1.21}$$

and $(\cdot,\cdot)$ is the standard inner product in $L_2(0,1)$.

To solve the problem numerically, we use the Petrov-Galerkin finite element method. To begin with, we describe some spaces and estimates which are necessary for the investigation of convergence.

We introduce a trial space $S_h \in H^1(\Omega)$ with a basis $\{\varphi_j\}_{j=0}^{M+1}$ and a test space $T_h \in H_0^1(\Omega)$ with a basis $\{\psi_j\}_{j=1}^{M}$. Let $a^h(\cdot,\cdot) : S_h \times T_h \to R$ be a bilinear form which approximates the form $a(\cdot,\cdot)$ and $f_h : T_h \to R$ be a functional which approximates the inner product $(f,\cdot)$. Then we have the following formulation of the Petrov-Galerkin method (see, for example, [39]): *find* $u^h \in S_h$ *satisfying the boundary conditions* (1.3) *and the equality*

$$a^h(u^h,v^h) = f_h(v^h) \quad \forall\, v^h \in T_h. \tag{1.22}$$

Since

$$S_h = span\{\varphi_0, ..., \varphi_{M+1}\}, \quad T_h = span\{\psi_1, ..., \psi_M\}$$

the formulation (1.22) is equivalent to the linear system of algebraic equations

$$L^h U^h = F^h \tag{1.23}$$

where $U^h = (u_1, ..., u_M)^T$ is the vector of unknowns, and $F^h = (f_h(\psi_1) - a(\varphi_0,\varphi_1)u_0, f_h(\psi_2), \ldots, f_h(\psi_{M-1}), f_h(\psi_M) - a(\varphi_{M+1},\varphi_M)u_{M+1})^T$ is the right-hand side vector. L^h is the matrix with the elements

$$L_{ij}^h = a^h(\varphi_j,\psi_i), \qquad i,j = 1, ..., M. \tag{1.24}$$

The usual way to investigate the convergence of (1.22) consists in evaluating the difference $u - u^h$ in the energy norm in terms of $u - u^I$ where u^I is the interpolant of the solution in S_h. Then we obtain the estimate in L_p–norm. But for the singularly perturbed problems the estimate of $u - u^I$ may be very poor because of the boundary layer component of the solution. Therefore we study the difference $u - u^h$ directly:

$$\begin{aligned} |a^h(u^h - u^I, w^h)| &= |a^h(u^h, w^h) - a^h(u^I, w^h) + a(u, w^h) - a(u, w^h)| \\ \leq &\quad |f^h(w^h) - f(w^h)| + |a(u, w^h) - a^h(u^I, w^h)| \qquad (1.25) \\ + &\quad |a(\rho, w^h) - a^h(\rho^I, w^h)| + |a(z_1, w^h) - a^h(z_1^I, w^h)| \quad \forall\, w^h \in T_h. \end{aligned}$$

Here u is the solution of the differential problem (1.1), (1.3), u^h is the solution of the discrete problem (1.22), and u^I, v_0^I, ρ^I, $z_1^I \in S_h$ are the interpolants of u, v_0, ρ, z_1 respectively.

The basic idea of the method discussed here is to the construct an approximation of the bilinear form that reduces the error of the boundary layer component. The general analysis of the problem and the construction of the discrete analogue which gives the first order ε–uniform convergence can be found in [121]. We cite some results from this work.

For vectors $V^h = (v_1, \cdots, v_M)^T \in \mathbf{R}^M$ we introduce the discrete p-norms

$$||V^h||_p = \begin{cases} \left(\sum\limits_{i=1}^{M} d_i |v_i|^p \, dx \right)^{1/p}, & 1 \leq p < \infty, \\ \max\limits_{1 \leq i \leq M} |v_i|, & p = \infty \end{cases} \qquad (1.26)$$

and

$$|||V^h|||_p = ||(D^h)^{-1} (L^h)^T V^h||_p, \quad 1 \leq p \leq \infty. \qquad (1.27)$$

Here D^h is the diagonal matrix with the positive elements

$$d_i = \text{meas}(\text{supp } \varphi_i), \quad i = 1, ..., M.$$

Note that (1.27) is a norm in $\mathbf{R}^M$ when the matrix L^h is invertible. Together with the space S_h we consider the space $\overset{\circ}{S}_h = span\{\varphi_1, ..., \varphi_M\}$. The spaces $\overset{\circ}{S}_h$ and T_h are equipped with different norms. In order to introduce these norms we use the isomorphisms $\overset{\circ}{S}_h \leftrightarrow \mathbf{R}^M$ and $T_h \leftrightarrow \mathbf{R}^M$ defined by

$$v^h = \sum_{i=1}^{M} v_i \varphi_i \in \overset{\circ}{S}_h, \quad V^h = (v_1, \cdots, v_M)^T \in \mathbf{R}^M,$$

$$w^h = \sum_{i=1}^{M} w_i \psi_i \in T_h, \quad W^h = (w_1, \cdots, w_M)^T \in \mathbf{R}^M.$$

We introduce for $v^h \in \overset{\circ}{S}_h$ and $w^h \in T_h$ the norms

$$\|v^h\|_{p,h} = \|V^h\|_p \qquad \text{and} \qquad |\!|\!|w^h|\!|\!|_{q,h} = |\!|\!|W^h|\!|\!|_q, \tag{1.28}$$

respectively.

Because of definitions (1.24), (1.28), and the Hőlder inequality the following estimate of the bilinear form a^h holds.

Lemma 4. [121] *Suppose that* $1 \le p \le \infty$, $1/p + 1/q = 1$ *and the matrix* L^h *is nonsingular. Then for all* $v^h \in \overset{\circ}{S}_h$ *and* $w^h \in T_h$ *we have*

$$|a^h(v^h, w^h)| \le \|v^h\|_{p,h} |\!|\!|w^h|\!|\!|_{q,h}. \tag{1.29}$$

Now we formulate the basic convergence result.

Theorem 5. *Let* u *and* u^h *be the solutions of the problems* (1.20) *and* (1.22), *respectively. Then for the interpolant* u^I *of* u *in* S_h *the error estimate*

$$\|u^I - u^h\|_{p,h} \le \sup_{w^h \in T_h/\{0\}} \frac{|(f, w^h) - f_h(w^h) + a^h(u^I, w^h) - a(u, w^h)|}{|\!|\!|w^h|\!|\!|_{q,h}} \tag{1.30}$$

holds where $1/q + 1/p = 1$, $1 \le p \le \infty$.

Remark. The error estimate in the continuous L^p–norm follows from the norm equivalence

$$c_8\|v\|_{p,h} \le \|v\|_p \le c_9\|v\|_{p,h} \quad \forall\, v \in \overset{\circ}{S}_h \tag{1.31}$$

where constants c_8, c_9 are independent of h.

The discrete problem which is first-order convergent, uniformly in ε, was constructed in [121]. In the next two sections we will obtain the second-order method.

1.2 The Finite Element Method with a Linear Quadrature Rule

In this section we introduce the restriction

$$b'(x) \ge 0 \quad \text{on} \quad [0, 1] \tag{1.32}$$

which simplifies the proof of stability. This restriction provides the fulfillment of the maximum principle for the problem (1.1) – (1.3). We show that for small ε this restriction can be introduced without loss of generality. Assume that for some $x_0 \in [0,1]$ we have $b'(x) < 0$. Introduce a new unknown function

$$w(x) = u(x)\exp(-\sigma x)$$

with the positive constant

$$\sigma = 1 + \max_{x\in[0,1]} \frac{|b'| - b'}{2b}.$$

Then the problem (1.1) – (1.3) is equivalent to the following one

$$\begin{aligned} -\varepsilon w'' + (b - 2\varepsilon\sigma)w' + (b' + b\sigma - \varepsilon\sigma^2)w &= f\exp(-\sigma x) \quad \text{on} \quad (0,1),\\ w(0) = u_0, \qquad w(1) &= u_1\exp(-\sigma). \end{aligned}$$

For small ε the coefficient of w is positive on the segment $[0,1]$ since $b' + b\sigma - \varepsilon\sigma^2 \geq b - \varepsilon\sigma^2 \geq 0$. Hence the maximum principle holds.

We consider the asymptotic expansion (1.11).

1.2.1 Construction of the quadrature rule

To approximate the solution u, we use the piecewise linear finite elements on a nonuniform grid

$$\overline{\omega}_h = \{x_i :\ i = 0,1,\ldots,n;\ 0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1\} \tag{1.33}$$

with a mesh size $h_i = x_i - x_{i-1}$. For simplicity we consider a quasiuniform grid satisfying the condition

$$c_{10}h \leq h_i \leq h = \max_{1\leq i\leq n} h_i. \tag{1.34}$$

We denote the set of interior nodes by

$$\omega_h = \{x_i :\ x_i \in \overline{\omega}_h,\ \ i = 1,\ldots,\ n-1\}.$$

Introduce the basis functions $\varphi_i(x) \in C[0,1]$ defined by

$$\varphi_i(x) = \begin{cases} (x - x_{i-1})/h_i, & \text{if} \quad x \in [x_{i-1}, x_i] \cap [0,1];\\ (x_{i+1} - x)/h_{i+1}, & \text{if} \quad x \in (x_i, x_{i+1}] \cap [0,1];\\ 0 & \text{otherwise} \end{cases}$$

and the spaces of trial and test functions

$$S_h = \operatorname{span}\{\varphi_0,\ldots,\varphi_n\} \quad \text{and} \quad T_h = \operatorname{span}\{\varphi_1,\ldots,\varphi_{n-1}\}.$$

To approximate the problem (1.20), we use the Petrov-Galerkin method: *find $u^h \in S_h$ such that $u^h(0) = u_0$, $u^h(1) = u_1$ and*

$$a(u^h, w^h) = (f, w^h) \quad \forall\, w^h \in T_h. \tag{1.35}$$

This approach has some disadvantages. With the boundary layer, the solution of the algebraic system has unsatisfactory accuracy. Besides, this system becomes unstable for $\varepsilon \le h$. Finally, when constructing the algebraic system, one have to integrate functions. Thus, the application of quadrature rules is quite natural. We choose quadrature rules in a special way to ensure stability and to improve the accuracy of the approximate problem obtained.

Therefore we return to the bilinear form (1.21). The first term is integrated exactly for any $u \in S_h$, $v \in T_h$. For the second term we use the following quadrature rule on each interval:

$$\int_{x_{i-1}}^{x_i} bv\, dx \approx (\alpha_i b_{i-1} v_{i-1} + \beta_i b_i v_i)\, h_i \tag{1.36}$$

where $v_i = v(x_i)$ for an arbitrary function $v(x)$. Using this formula for a we obtain the new bilinear form a_h of an algebraic type for $v \in S_h$ and $w^h \in T_h$:

$$a^h(v, w^h) = \sum_{i=1}^{n} (\varepsilon(v_i - v_{i-1})/h_i - \alpha_i b_{i-1} v_{i-1} - \beta_i b_i v_i)\,(w_i^h - w_{i-1}^h). \tag{1.37}$$

The standard way to justify the accuracy of the Galerkin solution is to use Strang's first lemma and the closeness of the bilinear forms a and a^h with arguments from the class of admissible functions. Unfortunately, in our case this method yields poor estimates due to the boundary layer components ρ_0, ρ_1. Therefore we choose the parameters α_i, β_i in such a way as to make these bilinear forms as close as possible just for the functions ρ_0, ρ_1. For example, for the function ρ_0 the exact equality

$$\int_{x_{i-1}}^{x_i} b\rho_0\, dx = (\alpha_i b_{i-1} \rho_{0,i-1} + \beta_i b_i \rho_{0,i})\, h_i$$

should be taken. However, this condition contains the integral in the left-hand side, that does not permit to obtain the explicit expression in the general case. Therefore for convenience we use (1.7), (1.10) for ρ_0 in the right-hand side of this equality and replace $b(x)$ by its linear interpolant. As a result, we arrive at the equality

$$\begin{aligned} &\int_{x_{i-1}}^{x_i} (b_{i-1}(x_i - x)/h_i + b_i(x - x_{i-1})/h_i) \exp(-b(1)(1-x)/\varepsilon)\, dx \\ &= \alpha_i b_{i-1} \exp(-b(1)(1 - x_{i-1})/\varepsilon) h_i + \beta_i b_i \exp(-b(1)(1 - x_i)/\varepsilon) h_i. \end{aligned} \tag{1.38}$$

Taking the integral in the left-hand side and dividing the obtained equality by $h_i \exp(-b(1)(1-x_i)/\varepsilon)$ we get

$$\begin{aligned} \alpha_i b_{i-1} \exp(-\sigma_i) + \beta_i b_i = b_{i-1} \left(\frac{1}{\sigma_i^2} - \frac{1}{\sigma_i} \exp(-\sigma_i) - \frac{1}{\sigma_i^2} \exp(-\sigma_i) \right) \\ + b_i \left(\frac{1}{\sigma_i} - \frac{1}{\sigma_i^2} + \frac{1}{\sigma_i^2} \exp(-\sigma_i) \right) \end{aligned} \tag{1.39}$$

where $\sigma_i = b(1)h_i/\varepsilon$. To the above equality we add the equation

$$\alpha_i + \beta_i = 1 \tag{1.40}$$

which permits to approximate an integral of a smooth function with the first-order accuracy. Thus, we arrive at the system of linear algebraic equations in two unknowns. Its determinant is given by

$$\xi_i = b_i - b_{i-1} \exp(-\sigma_i). \tag{1.41}$$

Since $b' \geq 0$ and $\exp(-\sigma_i) < 1$, ξ_i is strictly positive. Hence the system (1.39) – (1.40) has an unique solution.

Thus, we can expect that the boundary layer function ρ_0 satisfies the equality

$$\int_{x_{i-1}}^{x_i} b\rho_0 \, dx = (b\rho_0)_i^* h_i + O(h^3). \tag{1.42}$$

The proof of this statement is given later. Here we use the notation

$$(b\rho_0)_i^* = \alpha_i b_{i-1} \rho_{0,i-1} + \beta_i b_i \rho_{0,i}.$$

It is easy to verify that for ρ_1 we have

$$\int_{x_{i-1}}^{x_i} b\rho_1 \, dx = O(\varepsilon).$$

Actually the contribution of this term is still smaller due to the coefficient ε of the function ρ_1 in the expansion (1.11).

Now consider the remaining part of the solution

$$g(x) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 z(x). \tag{1.43}$$

For $g(x)$ the quadrature rule (1.36) has only the first-order accuracy:

$$\begin{aligned} \int_{xi-1}^{x_i} b(x)g(x) \, dx &= (bv_0)_i^* h_i + (1/2 - \beta_i) h_i^2 (bv_0)'_{i-1} \\ &\quad + (bv_1)_i^* h_i + O(h^3 + \varepsilon h^2 + \varepsilon^2 h) \\ &= (bg)_i^* h_i + (1/2 - \beta_i) h_i^2 (bg)'_{i-1} + O(h^3 + \varepsilon h^2 + \varepsilon^2 h). \end{aligned} \tag{1.44}$$

Here we use the fact that the functions v_0'', v_1', and z are bounded on $[0,1]$. Take into consideration the equality

$$-\varepsilon(\rho_0+\varepsilon\rho_1)''+(b(\rho_0+\varepsilon\rho_1))'=O(\varepsilon)$$

which results from the definition of ρ_0 and ρ_1. Then we transform the main term of the error to the form

$$(bg)'=(bg)'-\varepsilon g''+O(\varepsilon)=(bu)'-\varepsilon u''+O(\varepsilon)=f+O(\varepsilon).$$

Then instead of (1.44) we get

$$\begin{aligned}\int_{x_{i-1}}^{x_i} b(x)g(x)\,dx &= (bg)_i^* h_i \\ &\quad + (1/2-\beta_i)h_i^2 f_{i-1} + O(h^3+\varepsilon h^2+\varepsilon^2 h).\end{aligned} \tag{1.45}$$

When constructing the bilinear forms a and a^h all the terms are multiplied by $-(w^h)'$. Therefore the main term of the error $a(g,w^h)-a^h(g,w^h)$ on the segment $[x_{i-1},x_i]$ takes the form

$$-(1/2-\beta_i)h_i^2 f_{i-1}(w^h)'_{i-1/2}. \tag{1.46}$$

We construct the quadrature rule for the right-hand side to eliminate this term. We rewrite the functional in the right-hand side as

$$\int_0^1 f(x)w^h(x)\,dx=-\int_0^1 F(x)(w^h(x))'\,dx \quad \forall\, w^h\in T^h \tag{1.47}$$

with the antiderivative $F'(x)=f(x)$. Using the Taylor expansion, in a similar way as (1.44) we obtain

$$\int_{x_{i-1}}^{x_i} F(x)\,dx=F_i^* h_i+(1/2-\beta_i)h_i^2 f_{i-1}+O(h^3). \tag{1.48}$$

Thus, the main term of the error coincides with (1.44). In order to avoid the calculation of the antiderivative, we use the difference analogue of integration by parts taking into account the boundary conditions $w^h(0)=w^h(1)=$

0:

$$\int_0^1 f(x)w^h(x)\,dx = -\int_0^1 F(x)(w^h(x))'\,dx = -\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} F(x)(w^h)'\,dx$$

$$= -\sum_{i=1}^{n}(w^h)'_{i-1/2}(F_i^* h_i + (1/2-\beta_i)h_i^2 f_{i-1}) + O(h^3)\sum_{i=1}^{n}(w^h)'_{i-1/2}$$

$$= \sum_{i=1}^{n-1} w_i^h(F_{i+1}^* - F_i^*) - \sum_{i=1}^{n}(1/2-\beta_i)h_i^2 f_{i-1}(w^h)'_{i-1/2}$$

$$+O(h^3)\sum_{i=1}^{n}(w^h)'_{i-1/2}.$$

We choose the weights μ_i and ν_i in such a way as to replace the difference between the values of the antiderivative F by the function f with the third-order accuracy:

$$F_{i+1}^* - F_i^* = \mu_i f_{i-1} + \nu_i f_i + O(h^3). \tag{1.49}$$

Then we use the Taylor expansion at the point x_{i-1} and set the coefficients of h_i and h_i^2 to be equal:

$$\begin{aligned} \mu_i + \nu_i &= h_i(1-\beta_i) + h_{i+1}\beta_{i+1}, \\ 2\nu_i h_i &= h_i^2(1-\beta_i) + h_{i+1}\beta_{i+1}(2h_i + h_{i+1}). \end{aligned} \tag{1.50}$$

Hence μ_i and ν_i are uniquely determined. As a result, in the right-hand side we get the approximate functional

$$f_h(w^h) = \sum_{i=1}^{n}(\mu_i f_{i-1} + \nu_i f_i)w_i^h \tag{1.51}$$

with the coefficients μ_i and ν_i from (1.50).

On substitution of the bilinear form $a(\cdot,\cdot)$ and the right-hand side (f, w^h) into (1.35), we obtain the discrete problem: *find* $u^h \in S_h$ *such that* $u^h(0) = u_0$, $u^h(1) = u_1$, *and*

$$a^h(u^h, w^h) = f_h(w^h) \qquad \forall\, w^h \in T_h. \tag{1.52}$$

We rewrite this problem in the equivalent matrix-vector form: *construct the function*

$$u^h = \sum_{i=0}^{n}\gamma_i\varphi_i$$

with the weights γ_i *which satisfying the conditions* $\gamma_0 = u_0$, $\gamma_n = u_1$ *as well as the system of linear algebraic equations*

$$A^h \gamma = F^h \tag{1.53}$$

with the vector of the unknowns $\gamma = (\gamma_1, ..., \gamma_{n-1})^T$ and the given the right-hand side $F^h = (F_1^h, ..., F_{n-1}^h)^T$ where

$$\begin{aligned}
F_1^h &= \mu_1 f_0 + \nu_1 f_1 + a_1 u_0, \\
F_i^h &= \mu_i f_{i-1} + \nu_i f_i, \qquad\qquad i = 2, ..., n-2, \\
F_{n-1}^h &= \mu_{n-1} f_{n-2} + \nu_{n-1} f_{n-1} + e_{n-1} u_1.
\end{aligned}$$

The matrix A^h has the tridiagonal form

$$A^h = \begin{pmatrix}
d_1 & -e_1 & & & & & \\
-a_2 & d_2 & -e_2 & & & 0 & \\
 & \cdots & \cdots & \cdots & & & \\
 & & \cdots & \cdots & \cdots & & \\
 & & & \cdots & \cdots & \cdots & \\
 & 0 & & & -a_{n-2} & d_{n-2} & -e_{n-2} \\
 & & & & & -a_{n-1} & d_{n-1}
\end{pmatrix}$$

and its elements are given by

$$\begin{aligned}
a_i &= \varepsilon / h_i + \alpha_i b_{i-1}, \\
d_i &= \varepsilon / h_i + \varepsilon / h_{i+1} + \alpha_{i+1} b_i - \beta_i b_i, \\
e_i &= \varepsilon / h_{i+1} - \beta_{i+1} b_{i+1}, \quad i = 1, ..., n-1.
\end{aligned} \tag{1.54}$$

1.2.2 Properties of the discrete problem

Now we investigate the discrete problem.

Lemma 6. *Under the restrictions* (1.2), (1.4), (1.32) *for any* ε, $h > 0$ *the matrix* A^h *of the system* (1.53) *is an* M*-matrix and hence is nonsingular.*

Proof. First we consider the parameter α_i of the quadrature rule. From the equations (1.39), (1.40) we have

$$\begin{aligned}\alpha_i = {} & \frac{1}{b_i - b_{i-1}\exp(-\sigma_i)}\Big(b_{i-1}\frac{\sigma_i\exp(-\sigma_i) - 1 + \exp(-\sigma_i)}{\sigma_i^2} \\ & + b_i\,\frac{\sigma_i^2 - \sigma_i + 1 - \exp(-\sigma_i)}{\sigma_i^2}\Big) \\ = {} & \frac{1}{b_i - b_{i-1}\exp(-\sigma_i)}\Big(\frac{1-\exp(-\sigma_i)}{\sigma_i^2}\,(b_i - b_{i-1}) + b_i\Big) - \frac{1}{\sigma_i}\end{aligned} \tag{1.55}$$

where $\sigma_i = b(1)h_i/\varepsilon$. Since the inequalities $\sigma_i > 0$, $\exp(-\sigma_i) < 1$ and $b'(x) \geq 0$ hold for arbitrary ε and h, we have

$$\alpha_i > 1/\sigma_i \quad \forall\, \varepsilon, h_i > 0. \tag{1.56}$$

Due to (1.54), (1.56), and inequality $b_{i-1} \leq b(1)$ the estimate

$$0 \leq \frac{\varepsilon}{h_i} - \frac{b_{i-1}\varepsilon}{b(1)h_i} < \frac{\varepsilon}{h_i} + \alpha_i b_{i-1} = a_i$$

holds. Hence the coefficients a_i of the system (1.53) are strongly positive.

The proof of the positiveness of e_i is rather complicated. Because of the definition of β_{i+1} the following equalities hold:

$$\begin{aligned}e_i = {} & \frac{\varepsilon}{h_i} - \frac{b_{i+1}}{b_{i+1} - b_i\exp(-\sigma_i)}\Bigg(b_i\left(\frac{1}{\sigma_i^2} - \frac{\exp(-\sigma_i)}{\sigma_i} - \frac{\exp(-\sigma_i)}{\sigma_i^2} - \exp(-\sigma_i)\right) \\ & + b_{i+1}\left(\frac{1}{\sigma_i} - \frac{1}{\sigma_i^2} + \frac{\exp(-\sigma_i)}{\sigma_i^2}\right)\Bigg) \\ = {} & \frac{\varepsilon}{h_i} - \frac{b_{i+1}^2}{\sigma_i\left(b_{i+1} - b_i\exp(-\sigma_i)\right)} + \frac{b_{i+1}}{b_{i+1} - b_i\exp(-\sigma_i)}\Bigg(\frac{b_i\exp(-\sigma_i)}{\sigma_i} \\ & + b_i\exp(-\sigma_i) + \frac{b_{i+1} - b_i}{\sigma_i^2}\left(1 - \exp(-\sigma_i)\right)\Bigg).\end{aligned} \tag{1.57}$$

The difference of the two list terms in the right-hand side equals

$$\begin{aligned}& \frac{b(1)b_{i+1} - b_{i+1}^2 + b(1)b_i\exp(-\sigma_i)}{\sigma_i\left(b_{i+1} - b_i\exp(-\sigma_i)\right)} \\ & \qquad = \frac{b_{i+1}(b(1) - b_{i+1}) + b(1)b_i\exp(-\sigma_i)}{\sigma_i\left(b_{i+1} - b_i\exp(-\sigma_i)\right)}.\end{aligned} \tag{1.58}$$

Under the restrictions (1.2) and (1.32) both terms in the numerator are nonnegative and the denominator is positive. For the same reason, in the last term all three quantities in parentheses are nonnegative and the coefficient of the parenthetical expression is positive. Hence we have $e_i > 0 \quad \forall\, i = 1, ..., n-1$.

For the system (1.53) the following relations hold:

$$\begin{gathered} d_1 - a_2 > 0, \\ d_i - a_{i+1} - e_{i-1} = 0, \quad i = 2, ..., n-2, \\ d_{n-1} - e_{n-2} > 0. \end{gathered} \tag{1.59}$$

Because of the positiveness of a_i and e_i for all i, the matrix in (1.53) is diagonal-dominant along columns and strictly diagonal-dominant along the first and the last ones. Taking into account the fact that the matrix A^h is irreducible [21], this leads to the conclusion of Lemma 6. □

Lemma 7. *Let $W^h = (w_1, ..., w_{n-1})^T$ be the solution of the problem*

$$\left(A^h\right)^T W^h = Q^h \tag{1.60}$$

with some right-hand side $Q^h = (q_1, ..., q_{n-1})^T$. Under the restrictions (1.2), (1.4), (1.32), *and*

$$\varepsilon \le c_{11} h, \qquad c_{11} > 0 \tag{1.61}$$

the estimate

$$\sum_{i=2}^{n-1} |w_i - w_{i-1}| + |w_1| + |w_{n-1}| \le c_{12} \sum_{i=1}^{n-1} |q_i| \tag{1.62}$$

holds with a constant c_{12} independent of ε and h.

Proof. In a similar way as Lemma 3.3 in [121] we rewrite the system (1.60) in the form

$$\begin{gathered} e_{i-1}(w_i - w_{i-1}) + a_{i+1}(w_i - w_{i+1}) = q_i, \qquad i = 1, ..., n-1, \\ w_0 = w_n = 0. \end{gathered} \tag{1.63}$$

Using the notation

$$v_i = w_i - w_{i-1}$$

we write the difference equation from (1.63) as

$$v_{i+1} = \frac{e_{i-1}}{a_{i+1}} v_i - \frac{q_i}{a_{i+1}}.$$

Let us set

$$\prod_{k=l}^{l-1} \delta_k = 1 \quad \text{and} \quad \sum_{k=l}^{l-1} \delta_k = 0$$

for arbitrary δ_k. Then for all $i = 1, ..., n$ the equality

$$v_i = v_1 \prod_{j=1}^{i-1} \frac{e_{j-1}}{a_{j+1}} - \sum_{j=1}^{i-1} \left(\prod_{k=j+1}^{i-1} \frac{e_{k-1}}{a_{k+1}} \right) \frac{q_j}{a_{j+1}} \tag{1.64}$$

holds. Taking into account the equality

$$\sum_{j=1}^{n} v_j = w_n - w_0 = 0,$$

we obtain the initial value

$$v_1 = \sum_{i=1}^{n} \sum_{j=1}^{i-1} \left(\prod_{k=j+1}^{i-1} \frac{e_{k-1}}{a_{k+1}} \right) \frac{q_j}{a_{j+1}} \Big/ \sum_{i=1}^{n} \prod_{j=1}^{i-1} \frac{e_{j-1}}{a_{j+1}}. \tag{1.65}$$

From (1.64) and (1.65) we get

$$\begin{aligned} \sum_{i=1}^{n} |v_i| &\leq \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} \frac{e_{j-1}}{a_{j+1}} \right) |v_1| + \sum_{i=1}^{n} \sum_{j=1}^{i-1} \left(\prod_{k=j+1}^{i-1} \frac{e_{k-1}}{a_{k+1}} \right) \frac{|q_j|}{a_{j+1}} \\ &\leq 2 \sum_{i=1}^{n} \sum_{j=1}^{i-1} \left(\prod_{k=j+1}^{i-1} \frac{e_{k-1}}{a_{k+1}} \right) \frac{|q_j|}{a_{j+1}}. \end{aligned} \tag{1.66}$$

Taking into consideration the definition of the coefficients a_k and e_k in (1.54), the restriction (1.56), and the fact that b is bounded due to (1.2), we obtain the inequalities

$$\begin{aligned} e_{k-1} &= \varepsilon/h_i - \beta_k b_k \leq \varepsilon/c_{10}h, \\ a_{k+1} &= \varepsilon/h_i + \alpha_{k+1} b_k \geq \varepsilon/c_{10}h_i + B_0/2, \\ 1/a_{j+1} &\leq 1/\left(\varepsilon/h_i + B_0/2\right) \leq 2/B_0. \end{aligned}$$

Applying them to the right-hand side of (1.66), we have the estimate

$$\left(\prod_{k=j+1}^{i-1} \frac{e_{k-1}}{a_{k+1}} \right) \frac{|q_j|}{a_{j+1}} \leq \frac{2}{B_0} \eta^{i-j-1} |q_j| \quad \text{where} \quad \eta = \frac{1}{1 + B_0 c_{10} h/2\varepsilon}.$$

Using this inequality we change the order of summution in the right-hand side of (1.66):

$$\frac{4}{B_0}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\eta^{i-1-j}|q_j| = \frac{4}{B_0}\sum_{l=1}^{n}\sum_{j=1}^{n-l}\eta^{l-1}|q_j|$$
$$\leq \frac{4}{B_0}\sum_{l=1}^{n}\eta^{l-1}\sum_{j=1}^{n-1}|q_j| \leq \frac{4}{B_0(1-\eta)}\sum_{j=1}^{n-1}|q_j|. \tag{1.67}$$

Thus we get (1.62) with the constant $c_{12} = 4(1 + 2c_{11}/B_0c_{10})/B_0$. □

In the terms of the norms in the spaces of trial and test functions (1.26) and (1.28) the estimate (1.62) for functions $w^h \in T_h$ has the form

$$\|(w^h)'\|_1 \leq c_{12}|\!|\!|w^h|\!|\!|_{1,h}. \tag{1.68}$$

1.2.3 Convergence result

Now we consider the main theorem of this section.

Theorem 8. *Let* (1.2), (1.4), (1.32), *and* (1.61) *be valid for the problems* (1.53) *and* (1.1) – (1.3) *with the solutions* u^h *and* u *respectively. Then the estimate*

$$\max_{0\leq i\leq n}|u_i^h - u_i| \leq c_{15}(h^2 + \varepsilon^2/h + \varepsilon h + \varepsilon^2) \tag{1.69}$$

holds.

We will proof the same theorem in more general case in the next section.

Notice that according to (1.69) the approximate solution has the second-order accuracy with respect to h for $\varepsilon << h$, in particular, for $\varepsilon \leq h^{3/2}$. The numerical experiments presented in Chapter 3 confirm this result. Thus, for $\varepsilon < h$ the constructed scheme is more accurate in comparison with other well-known methods, for example, with the first-order scheme from [121].

1.3 The Finite Element Method with Nonlinear Quadrature Rule

In this section the monotonicity (1.32) of the function $b(x)$ is not required because of the application of the nonlinear quadrature rule for the approximation of the convective term in the bilinear form. Theoretically this condition is not too restrictive, but in practice it is inconvenient, for example, when the function $b(x)$ is given discretely.

1.3.1 Construction of the quadrature rule

We return to the approximation of the bilinear form (1.21). The first term is integrated exactly for any $u \in S_h$, $v \in T_h$. For the second one we use the following quadrature rule on each interval:

$$\int_{x_{i-1}}^{x_i} bv\,dx \approx (\alpha_i b_{i-1} + \beta_i b_i)\,(\alpha_i v_{i-1} + \beta_i v_i)\,h_i. \tag{1.70}$$

Unlike the similar formula (1.36), in this case in the points for the calculation of the values of the functions b and v are choose individually with the help of the parameters α_i and β_i on each interval $[x_{i-1}, x_i]$. When we use (1.70) for a, we obtain the new bilinear form a^h of an algebraic type for $v, w^h \in S_h$:

$$\begin{aligned} a^h(v, w^h) = \sum_{i=1}^{n} \Big(&\varepsilon(v_i - v_{i-1})/h_i \\ &- (\alpha_i b_{i-1} + \beta_i b_i)\,(\alpha_i v_{i-1} + \beta_i v_i)\Big)(w_i^h - w_{i-1}^h). \end{aligned} \tag{1.71}$$

As before, we choose the parameters α_i, β_i so that the bilinear forms a and a^h are as close as possible just for the function ρ_0. Generally speaking, the exact equality

$$\int_{x_{i-1}}^{x_i} b\rho\,dx = (\alpha_i b_{i-1} + \beta_i b_i)\,(\alpha_i \rho_{0,i-1} + \beta_i \rho_{0,i})\,h_i$$

should be taken. However, this condition contains the integral in the left-hand side, that does not permit to obtain the explicit expression in the general case. Therefore for convenience we replace $b(x)$ by its value $b(x) \approx b_i^* = \alpha_i b_{i-1} + \beta_i b_i$ on $[x_{i-1}, x_i]$. Thus we arrive at the equality

$$\begin{aligned} &\int_{x_{i-1}}^{x_i} b_i^* \exp(-b_i^*(1-x)/\varepsilon)\,dx \\ &= b_i^*\big(\alpha_i \exp(-b_i^*(1-x_{i-1})/\varepsilon) + \beta_i \exp(-b_i^*(1-x_i)/\varepsilon)\big)h_i. \end{aligned} \tag{1.72}$$

Taking the integral in the left-hand side and dividing the obtained equality by $h_i \exp(-b_i^*(1-x_i)/\varepsilon)$, we get

$$\alpha_i \exp(-\sigma_i) + \beta_i = (1 - \exp(-\sigma_i))\,/\sigma_i \tag{1.73}$$

where $\sigma_i = b_i^* h_i/\varepsilon$. To the above equality we add the equation

$$\alpha_i + \beta_i = 1 \tag{1.74}$$

which permits to approximate an integral of a smooth function with the first-order accuracy. Thus, we arrive at the system of linear algebraic equations in two unknowns.

Notice that for the function b with the constant value $b_{const,i}$ on the segment $[x_{i-1}, x_i]$ the system (1.73) – (1.74) becomes linear:

$$\alpha_{const,i} \exp(-\sigma_i) + \beta_{const,i} = \frac{1}{\sigma_i} (1 - \exp(-\sigma_i)),$$
$$\alpha_{const,i} + \beta_{const,i} = 1.$$

Its solution is obtained in the same way as in [121]:

$$\alpha_{const,i} = \frac{1}{1 - \exp(-\sigma_i)} - \frac{1}{\sigma_i}, \quad \beta_{const,i} = \frac{1}{\sigma_i} - \frac{\exp(-\sigma_i)}{1 - \exp(-\sigma_i)}.$$

In particular, for any positive $b_{const,i}$ this solution satisfies the inequalities

$$1/2 < \alpha_{const,i} < 1, \quad 0 < \beta_{const,i} < 1/2 \quad \forall\ h_i, \varepsilon > 0.$$

Further we consider the case

$$\varepsilon \le h^2 \tag{1.75}$$

which is of practical importance. We express β_i from the system (1.73) – (1.74):

$$\beta_i = \frac{1}{\sigma_i} - \frac{\exp(-\sigma_i)}{1 - \exp(-\sigma_i)}. \tag{1.76}$$

From (1.74) it follows that

$$\alpha_i = 1 - \beta_i. \tag{1.77}$$

Taking into account the definition of b_i^* and (1.77), we can write σ_i as

$$\sigma_i = (b_{i-1} + \beta_i (b_i - b_{i-1}))\, h_i/\varepsilon.$$

Hence there exists at least one solution α_i, β_i of the system (1.73) – (1.74) with the properties

$$1/2 < \alpha_i < 1, \quad 0 < \beta_i < 1/2.$$

This follows from the fact that for $\beta_i = 0$ the left-hand side of (1.76) is smaller than the right-hand one, and the opposite is true for $\beta_i = 1$. Therefore, the root can be found by the bisection method.

Now consider the remaining part of the solution

$$g(x) = v_0(x) + \varepsilon z_1(x).$$

In a similar way as in (1.44), using the fact that the functions v_0'', v_1', and z_1 are bounded on $[0,1]$, we can show that the quadrature rule (1.70) has only the first-order accuracy:

$$\int_{x_{i-1}}^{x_i} b(x)g(x)\,dx = b_i^* g_i^* h_i + (1/2 - \beta_i)h_i^2 (bg)'_{i-1} + O(h_i^3). \qquad (1.78)$$

We calculate $(b(x)g(x))'$ in (1.78) only at the interior points of the domain Ω. Therefore, considering the inequality (1.75) and the definition of ρ_0 and ρ_1, we obtain the estimate

$$-\varepsilon\left(\rho_0 + \varepsilon\rho_1\right)'' + \left(b\left(\rho_0 + \varepsilon\rho_1\right)\right)' \le c_9\varepsilon \quad \text{for} \quad x \le 1 - h_n.$$

Taking into account this estimate, we transform the main term of the error in (1.78) as follows:

$$(bg)' = (bg)' - \varepsilon g'' + O(\varepsilon) = (bu)' - \varepsilon u'' + O(\varepsilon) = f + O(\varepsilon).$$

Then instead of (1.78) we get

$$\int_{x_{i-1}}^{x_i} b(x)g(x)\,dx = b_i^* g_i^* h_i + (1/2 - \beta_i)h_i^2 f_{i-1} + O(h_i^3 + \varepsilon h_i^2 + \varepsilon^2 h_i).$$

Recall that when constructing the bilinear forms a and a^h all the terms are multiplied by $-(w^h)'$. Therefore the main term of the error on the segment $[x_{i-1}, x_i]$ has the form

$$-(1/2 - \beta_i)h_i^2 f_{i-1}(w^h)'_{i-1/2}.$$

We use the same functional of the right-hand side as in (1.51) with the coefficients (1.50) to eliminate this term.

Substituting the bilinear form $a(\cdot,\cdot)$ and the right-hand side (f, w^h) into (1.35), we obtain the discrete problem: *find* $u^h \in S_h$ *such that* $u^h(0) = u_0$, $u^h(1) = u_1$, *and*

$$a^h(u^h, w^h) = f_h(w^h) \qquad \forall\, w^h \in T_h. \qquad (1.79)$$

We rewrite this problem in the equivalent matrix-vector form: *construct the function*

$$u^h = \sum_{i=0}^{n} \tau_i \varphi_i$$

with the weights τ_i which satisfy the conditions $\tau_0 = u_0$, $\tau_n = u_1$ as well as the system of linear algebraic equations

$$A^h \tau = F \tag{1.80}$$

with the vector of unknowns

$$\tau = (\tau_1, ..., \tau_{n-1})^T$$

and the given the right-hand side

$$F^h = (F_1^h, ..., F_{n-1}^h)^T$$

where

$$\begin{aligned} F_1^h &= \mu_1 f_0 + \nu_1 f_1 + a_1 u_0, \\ F_i^h &= \mu_i f_{i-1} + \nu_i f_i, \qquad\qquad i = 2, ..., n-2, \\ F_{n-1}^h &= \mu_{n-1} f_{n-2} + \nu_{n-1} f_{n-1} + e_{n-1} u_1. \end{aligned}$$

The matrix A^h has the tridiagonal form

$$A^h = \begin{pmatrix} d_1 & -e_1 & & & & & \\ -a_2 & d_2 & -e_2 & & & 0 & \\ & \cdots & \cdots & \cdots & & & \\ & & \cdots & \cdots & \cdots & & \\ & & & \cdots & \cdots & \cdots & \\ & 0 & & & -a_{n-2} & d_{n-2} & -e_{n-2} \\ & & & & & -a_{n-1} & d_{n-1} \end{pmatrix}$$

where

$$\begin{aligned} a_i &= \varepsilon/h_i + \alpha_i b_i^*, \\ d_i &= \varepsilon/h_i + \varepsilon/h_{i+1} + \alpha_{i+1} b_{i+1}^* - \beta_i b_i^*, \\ e_i &= \varepsilon/h_{i+1} - \beta_{i+1} b_{i+1}^*. \end{aligned} \tag{1.81}$$

1.3.2 Properties of the discrete problem

Now we investigate the discrete problem.

Lemma 9. *When the conditions* (1.2), (1.4), (1.75) *are satisfied for any h, $\varepsilon > 0$ the inequalities*

$$1/2 < \alpha_i < 1, \quad 0 < \beta_i < 1/2 \tag{1.82}$$

hold.

The lemma is proved in the same way as Lemma 3.2 from [121].

Lemma 10. *When the conditions* (1.2), (1.4), (1.75) *are satisfied for any* h, $\varepsilon > 0$ *the matrix* A^h *of the system* (1.80) *is an* M*-matrix and hence is nonsingular.*

Proof. From (1.81), (1.82), and (1.2) it follows that $a_i > 0$ for all i. We show that $e_i > 0$ for all i. For e_i we have

$$e_i = \frac{\varepsilon}{h_{i+1}} - \beta_{i+1} b^*_{i+1}.$$

From (1.76) it follows that

$$\beta_{i+1} = \frac{1}{\sigma_{i+1}} - \frac{\exp(-\sigma_{i+1})}{1 - \exp(-\sigma_{i+1})}.$$

Collecting two last equalities and the definition $\sigma_{i+1} = b^*_{i+1} h_{i+1}/\varepsilon$, we get

$$e_i = \frac{b^*_{i+1}}{1 - \exp(-\sigma_{i+1})}.$$

Since the inequalities $\sigma_i > 0, \quad \exp(-\sigma_i) < 1$, and $b(x) \geq B_0 > 0$ hold for any ratio between ε and h, we have

$$e_i > 0 \quad \forall\, \varepsilon, h > 0.$$

For the system (1.80) the relations

$$\begin{aligned} d_1 - a_2 &> 0, \\ d_i - a_{i+1} - e_{i-1} &= 0, \quad i = 2, ..., n-2, \\ d_{n-1} - e_{n-2} &> 0 \end{aligned}$$

hold. Because a_i and e_i are positive for all i, the matrix A^h is diagonal-dominant along columns and strongly diagonal-dominant along the first and last ones. Taking into account the fact that the matrix A^h is irreducible [21], this completes the proof. □

Lemma 11. *Let* $W^h = (w_1, ..., w_{n-1})^T$ *be the solution of the problem*

$$\left(A^h\right)^T W^h = Q^h \tag{1.83}$$

with some right-hand side $Q^h = (q_1, ..., q_{n-1})^T$. *Under the restrictions* (1.2), (1.4), *and* (1.75) *the estimate*

$$\sum_{i=2}^{n-1} |w_i - w_{i-1}| + |w_1| + |w_{n-1}| \leq c_{10} \sum_{i=1}^{n-1} |q_i| \tag{1.84}$$

holds with a constant c_{10} *independent of* ε *and* h.

Proof. We use the inequality (1.66) from Lemma 7:

$$\sum_{i=1}^{n} |v_i| \le 2 \sum_{i=1}^{n} \sum_{j=1}^{i-1} \left(\prod_{k=j+1}^{i-1} \frac{e_{k-1}}{a_{k+1}} \right) \frac{q_j}{a_{j+1}}. \tag{1.85}$$

Taking into consideration the definition of the coefficients a_k and e_k in (1.81) and the fact that b is bounded, in accordance with (1.2) we get

$$\left(\prod_{k=j+1}^{i-1} \frac{e_{k-1}}{a_{k+1}} \right) \frac{1}{a_{j+1}} = \frac{1}{b_i^*} \left(1 - \exp(-\sigma_i)\right) \prod_{k=j+1}^{i-1} \exp(-\sigma_k)$$
$$\le \frac{1}{B_0} \left(1 - \exp(-B_1 h_i/\varepsilon)\right) \exp\left(-\frac{B_0}{\varepsilon}(x_{i-1} - x_j) \right).$$

Using the last inequality, we change the order of summation in (1.85):

$$\sum_{i=1}^{n} |v_i| \le \frac{2}{B_0} \sum_{i=1}^{n} \left(1 - \exp(-B_1 h_i/\varepsilon)\right) \sum_{j=1}^{i-1} \exp\left(-\frac{B_0}{\varepsilon}(x_{i-1} - x_j) \right) q_j$$
$$= \frac{2}{B_0} \sum_{j=1}^{n-1} q_j \sum_{i=j+1}^{n-1} \left(1 - \exp(-B_1 h_{i+1}/\varepsilon)\right) \exp\left(-\frac{B_0}{\varepsilon}(x_i - x_j) \right)$$
$$\le \frac{2}{B_0} \sum_{j=1}^{n-1} q_j \sum_{i=j+1}^{n-1} d \frac{\varepsilon}{x_i - x_j}.$$

Here we applied the inequalities $1 - \exp(-t) \le 1$ and $t \exp(-\alpha t) \le d$ which are valid for $t \in (0, 1)$ and $\alpha \ge 0$ with a constant d. Due to (1.75) and (1.34), the last sum over i can be estimated by a constant c_{11}. Taking into account the definition of v_i, we complete the proof of the estimate (1.84). $\square$

In terms of the norms in the spaces S_h and T_h the estimate (1.84) for functions $w^h \in T_h$ has the form

$$\|(w^h)'\|_1 \le c_9 \|\!| w^h |\!\|_{1,h}. \tag{1.86}$$

1.3.3 Convergence theorem

Now we consider the main result of this section.

Theorem 12. *Let u be the solution of the problem* (1.1), (1.3) *with the conditions* (1.2), (1.4), *and u^h be the solution of the problem* (1.80) *with the condition* (1.75). *Then the estimate*

$$\max_{0 \le i \le n} |u_i^h - u_i| \le c_{15}(h^2 + \varepsilon h + \varepsilon + \varepsilon^2 + \varepsilon^2/h) \tag{1.87}$$

holds.

Proof. By Theorem 5 for $p = \infty$ the estimate

$$\max_{0\le i\le n} |u_i^h - u_i| = \|u^h - u^I\|_{\infty,h} \le \sup_{w^h\in T_h} \frac{|(f,w^h) - f_h(w^h) + a^h(u^h,w^h) - a(u^h,w^h)|}{\|w^h\|_{1,h}} \tag{1.88}$$

holds.

We denote by $g(x)$ the sum of the smooth component and the remainder term in the expansion (1.11):

$$g(x) = v_0 + \varepsilon v_1 + \varepsilon^2 z(x).$$

Then we can write

$$\begin{aligned} |(f,w^h) - f_h(w^h) + a^h(u^h,w^h) - a(u^h,w^h)| &\le |a^h(\rho_0,w^h) - a(\rho_0,w^h)| \\ +\varepsilon|a^h(\rho_1,w^h) - a(\rho_1,w^h)| &+ |(f,w^h) - f_h(w^h) + a^h(g,w^h) - a(g,w^h)|. \end{aligned} \tag{1.89}$$

Consider the first term in the right-hand side

$$\Big|\sum_{i=1}^n \left(\varepsilon(\rho_{0,i} - \rho_{0,i-1})/h_i - b_i^*\rho_{0,i}^*\right)\left(w_i^h - w_{i-1}^h\right) - \int_0^1 (\varepsilon\rho_0' - b\rho_0)(w^h)'\,dx\Big|$$
$$= \Big|\int_0^1 b\rho_0(w^h)'\,dx - \sum_{i=1}^n b_i^*\rho_{0,i}^*\left(w_i^h - w_{i-1}^h\right)\Big|.$$

Rewrite the term

$$A_i = \frac{1}{h_i}\int_{x_{i-1}}^{x_i} b\rho_0\,dx - b_i^*\rho_{0,i}^*$$

as

$$\begin{aligned} A_i &= \frac{1}{h_i}\int_{x_{i-1}}^{x_i} \left(b\rho + b\left(\rho_0 - \rho\right)\right)\,dx - b_i^*\rho_i^* - b_i^*\left(\rho_{0,i}^* - \rho_i^*\right) \\ &\le \frac{1}{h_i}\int_{x_{i-1}}^{x_i} b\rho\,dx - b_i^*\rho_i^* + c_{16}\varepsilon = \tilde{A}_i. \end{aligned}$$

Here we use the estimate (1.19) from Lemma 3. Using the identity (1.72), we get

$$\begin{aligned} &b_i^*\left(\alpha_i \exp(-(1-x_{i-1})b_i^*/\varepsilon) + \beta_i \exp(-(1-x_i)b_i^*/\varepsilon)\right) \\ &\qquad = \frac{1}{h_i}\int_{x_{i-1}}^{x_i} b_i^* \exp(-(1-x)b_i^*/\varepsilon)\,dx. \end{aligned}$$

Then we transform $\tilde{A}_i$ to the form

$$\begin{aligned}\tilde{A}_i = {} & b_i^*\beta_i \left(\exp(-(1-x_i)b_i^*/\varepsilon) - \exp(-(1-x_i)b_i/\varepsilon)\right) \\ & + b_i^*\alpha_i \left(\exp(-(1-x_{i-1})b_i^*/\varepsilon) - \exp(-(1-x_{i-1})b_{i-1}/\varepsilon)\right) \\ & + \frac{1}{h_i}\int_{x_{i-1}}^{x_i} (b(x) - b_i^*)\,\rho(x)\,dx \\ & + \frac{b_i^*}{h_i}\int_{x_{i-1}}^{x_i} \left(\exp(-(1-x)b(x)/\varepsilon) - \exp(-(1-x)b_i^*/\varepsilon)\right)\,dx + c_{16}\varepsilon.\end{aligned} \tag{1.90}$$

The first term is estimated by the mean-value theorem:

$$\begin{aligned}|\exp(-(1-x_i)b_i^*/\varepsilon) - \exp(-(1-x_i)b_i/\varepsilon)| \quad & \le \\ |b_i^* - b_i|\,\frac{1-x_i}{\varepsilon}\exp(-(1-x_i)\tilde{b}/\varepsilon) \quad \text{where} \quad & \tilde{b} \in [B_0, B_1].\end{aligned} \tag{1.91}$$

Besides, $|b_i^* - b_i| \le h_i\|b'\|_\infty$ and the function $t^2\exp(-B_0 t)$ is bounded by a constant c_{17} on $(0,\infty)$. Using (1.2) and (1.82), we obtain

$$\begin{aligned}& b_i^*\beta_i\,|\exp(-(1-x_i)b_i^*/\varepsilon) - \exp(-(1-x_i)b_i/\varepsilon)| \\ & \le c_{17}B_1\beta_i\|b'\|_\infty\frac{\varepsilon h_i}{1-x_{i-1}} \le c_{18}\varepsilon.\end{aligned}$$

In a similar way we estimate the second term in (1.90):

$$b_i^*\alpha_i\,|\exp(-(1-x_{i-1})b_i^*/\varepsilon) - \exp(-(1-x_{i-1})b_{i-1}/\varepsilon)| \le c_{19}\varepsilon.$$

The third term is also estimated with the help of (1.2) and (1.82):

$$\begin{aligned}& \frac{1}{h_i}\left|\int_{x_{i-1}}^{x_i} (b(x) - b_i^*)\,\rho(x)\,dx\right| \le \|b'\|_\infty\int_{x_{i-1}}^{x_i} \exp(-(1-x)B_0/\varepsilon)\,dx \\ & = \|b'\|_\infty\frac{\varepsilon}{B_0}\left(\exp(-(1-x_i)B_0/\varepsilon) - \exp(-(1-x_{i-1})B_0/\varepsilon)\right) \le c_{20}\varepsilon.\end{aligned} \tag{1.92}$$

The integrand in the fourth term is estimated in the same way as in (1.91):

$$\begin{aligned}& |\exp(-(1-x)b(x)/\varepsilon) - \exp(-(1-x)b_i^*/\varepsilon)| \\ & \le h_i\|b'\|_\infty\frac{1-x}{\varepsilon}\exp(-(1-x)\tilde{\tilde{b}}/\varepsilon), \quad \text{where} \quad \tilde{\tilde{b}} \in [B_0, B_1].\end{aligned} \tag{1.93}$$

Using this inequality, we obtain the estimate of the fourth term in (1.90):

$$\begin{aligned}
&\left|\frac{b_i^*}{h_i}\int_{x_{i-1}}^{x_i}\left(\exp(-(1-x)b(x)/\varepsilon)-\exp(-(1-x)b_i^*/\varepsilon)\right)dx\right| \\
&\le c_{22}\int_{x_{i-1}}^{x_i}\frac{1-x}{\varepsilon}\exp(-(1-x)\tilde{\tilde{b}}/\varepsilon)\,dx \\
&\le c_{23}\varepsilon\left(1+\frac{(1-x)\tilde{\tilde{b}}}{\varepsilon}\right)\exp(-(1-x)\tilde{\tilde{b}}/\varepsilon)\Big|_{x_{i-1}}^{x_i} \\
&\le c_{24}\varepsilon\left(\exp(-(1-x_i)\tilde{\tilde{b}}/\varepsilon)-\exp(-(1-x_{i-1})\tilde{\tilde{b}}/\varepsilon)\right)\le c_{24}\varepsilon.
\end{aligned} \tag{1.94}$$

Summarizing the estimates (1.91)–(1.94), we can write

$$|A_i|\le c_{26}\varepsilon. \tag{1.95}$$

Thus, for the first term in (1.89) the following estimate holds:

$$\begin{aligned}
|a^h(\rho_0,w^h)-a(\rho_0,w^h)| &\le \left|\sum_{i=1}^n A_i(w_i-w_{i-1})\right|\le c_{27}\varepsilon\sum_{i=1}^n|w_i-w_{i-1}| \\
&\le c_{28}\varepsilon\|(w^h)'\|_{1,h}\le c_{29}\varepsilon|||(w^h)'|||_{1,h}\,.
\end{aligned} \tag{1.96}$$

Estimate the second term in (1.89):

$$\begin{aligned}
&\left|\sum_{i=1}^n\left(\varepsilon(\rho_{1,i}-\rho_{1,i-1})/h_i-b_i^*\rho_{1,i}^*\right)\left(w_i^h-w_{i-1}^h\right)-\int_0^1(\varepsilon\rho_1'-b\rho_1)(w^h)'\,dx\right| \\
&\qquad=\left|\int_0^1 b\rho_1(w^h)'\,dx-\sum_{i=1}^n b_i^*\rho_{1,i}^*\left(w_i^h-w_{i-1}^h\right)\right|.
\end{aligned}$$

Consider the expression

$$B_i=\frac{1}{h_i}\int_{x_{i-1}}^{x_i}b\rho_1\,dx-b_i^*\left(\alpha_i\rho_{1,i-1}+\beta_i\rho_{1,i}\right).$$

Taking into consideration the form of the function ρ_1, we can estimate the first term in the above expression:

$$\left|\frac{1}{h_i}\int_{x_{i-1}}^{x_i}b\rho_1\,dx\right|\le c_{30}\varepsilon/h_i.$$

The second term is bounded due to the estimates (1.2) and (1.82). Thus, we have

$$B_i \le c_{31}\varepsilon/h_i.$$

Hence, we obtain

$$\begin{aligned}|a^h(\rho_1, w^h) - a(\rho_1, w^h)| &\le \left|\sum_{i=1}^{n} B_i(w_i - w_{i-1})\right| \\ &\le \varepsilon c_{32}\sum_{i=1}^{n}\frac{1}{h_i}|w_i - w_{i-1}| \le \frac{\varepsilon}{h}c_{33}\|(w^h)'\|_{1,h} \le \frac{\varepsilon}{h}c_{34}|||(w^h)'|||_{1,h}.\end{aligned} \tag{1.97}$$

This estimate is worse than (1.96). However, the boundary layer component ρ_1 and the estimate (1.97) have to be multiplied by ε, that gives the same order of convergence.

Finally, consider the last term in (1.89):

$$\begin{aligned}&\Big|\int_0^1 fw^h\,dx - \sum_{i=1}^{n}(\mu_i f_{i-1} + \nu_i f_i)w_i^h - \int_0^1 (\varepsilon g' - bg)(w^h)'\,dx \\ &\quad - \sum_{i=1}^{n}\Big(\varepsilon(g_i - g_{i-1})/h_i - b_i^* g_i^*\Big)(w_i^h - w_{i-1}^h)\Big| \\ &\le \Big| - \int_0^1 F(x)(w^h)'\,dx + \int_0^1 bg(w^h)' - \\ &\quad - \sum_{i=0}^{n-1}\left(\left(\left(F_{i+1}^* - F_i^*\right) + \eta_i h_i^3\right) w_i^h + b_i^* g_i^*(w_i^h - w_{i-1}^h)\right) \Big| \\ &= \Big|\int_0^1 (bg - F)(w^h)' + \sum_{i=1}^{n}\left(F_i^*(w_i^h - w_{i-1}^h) + \eta_i h^3 w_i^h - b_i^* g_i^*(w_i^h - w_{i-1}^h)\right)\Big|\end{aligned}$$

where $F(x)$ is the antiderivative of $f(x)$ and η_i are the values of a function bounded on $[0, 1]$. Consider the term

$$\begin{aligned}C_i = \frac{1}{h_i}\int_{x_{i-1}}^{x_i}(bg - F)\,dx + \alpha_i F_{i-1} + \beta_i F_i \\ - (\alpha_i b_{i-1} - \beta_i b_i)(\alpha_i g_{i-1} - \beta_i g_i).\end{aligned}$$

Using the expansion of the functions bv_0, bv_1 and $b^*v_0^*$, $b^*v_1^*$ into the Taylor series at the point x_{i-1} and taking into account the fact that bz, b^*z^* are bounded, we get

$$\frac{1}{h_i}\int_{x_{i-1}}^{x_i} bg\,dx - b^*g^* = (1/2 - \beta_i)\,h_i(bg)'_{i-1} + O(h^2 + \varepsilon h + \varepsilon^2). \tag{1.98}$$

In view of the definitions of the boundary layer components ρ_0 and ρ_1 the estimate

$$-\varepsilon\,(\rho_0+\varepsilon\rho_1)''+(b\,(\rho_0+\varepsilon\rho_1))'\le c_{35}\varepsilon\quad\forall\,x\in[0,1]$$

holds. With the last inequality we have

$$(bg)'=(bg)'-\varepsilon g''+O(\varepsilon)=(bu)'-\varepsilon u''+O(\varepsilon)=f+O(\varepsilon).$$

Than the main term of the error in (1.98) has the form

$$-(1/2-\beta_i)h_i f_{i-1}(w^h)'_{i-1/2}+O(h^2+\varepsilon h).$$

From the identity (1.48) we obtain

$$-\frac{1}{h_i}\int_{xi-1}^{x_i}F(x)\,dx+F_i^*=-(1/2-\beta_i)h_i f_{i-1}+O(h^2).$$

Thus, the estimate

$$|C_i|\le c_{36}(h^2+\varepsilon h+\varepsilon^2)$$

holds and we have

$$\begin{aligned}
&|(f,w^h)-f_h(w^h)+a^h(g,w^h)-a(g,w^h)|\le\Big|\sum_{i=1}^{n}C_i(w_i^h-w_{i-1^h})+h_i^3\eta_i w_i\Big|\\
&\le c_{37}(h^2+\varepsilon h+\varepsilon^2)\sum_{i=1}^{n}|w_i^h-w_{i-1^h}|\le c_{38}(h^2+\varepsilon h+\varepsilon^2)\|(w^h)'\|_{1,h}\qquad(1.99)\\
&\le c_{39}(h^2+\varepsilon h+\varepsilon^2)|||w^h|||_{1,h}.
\end{aligned}$$

Finally, the estimate (1.87) follows from the relations (1.88), (1.89) and estimates (1.96), (1.97), and (1.99). □

Notice that, as before, for $\varepsilon\ll 1$ and even for $\varepsilon\le h$ in the case of practical importance, the accuracy of the obtained solution in accordance with the estimate (1.87) is of the second order with respect to h. This is confirmed by numerical experiments in Chapter 3.

As a result, for $\varepsilon<h$ the constructed scheme is more accurate than the similar one of the first-order accuracy from [121]. Therefore we concentrate our efforts upon this case. It should be noted that the convergence is proved for a non-uniform grid.

Now we discuss the question connecting the calculation of the coefficients α_i and β_i of the nonlinear system (1.73) – (1.74) on each interval $[x_{i-1},x_i]$.

Taking into account (1.76), we have the following formula for the coefficient β_i:

$$\beta_i = \frac{1}{\sigma_i} - \frac{\exp(-\sigma_i)}{1-\exp(-\sigma_i)}, \text{ where } \sigma_i = (b_{i-1} + \beta_i\,(b_i - b_{i-1}))\,h_i/\varepsilon.$$

Since the derivative b' is bounded, we have

$$b_i = b_{i-1} + h_{i-1}\delta_{i-1} \quad \text{where} \quad |\delta_{i-1}| \le \|b'\|_\infty.$$

Using this notation, from the system (1.73) – (1.74) we get

$$\begin{aligned} \beta_i = &-\frac{\exp(-\sigma_i)}{1-\exp(-\sigma_i)} \\ &+ \frac{1}{b_{i-1}+h_{i-1}\delta_{i-1}\beta_i}\left(\frac{b_{i-1}}{\sigma_{i-1}} + \frac{h_{i-1}\delta_{i-1}}{\sigma_{i-1}(1-\exp(-\sigma_{i-1}))} - \frac{h_{i-1}\delta_{i-1}}{\sigma_{i-1}^2}\right). \end{aligned} \tag{1.100}$$

Taking into account (1.2) and (1.75), we obtain the following estimate of the derivative of the right-hand side of (1.100) with respect to β_i:

$$\left|\frac{h_{i-1}\delta_{i-1}}{(b_{i-1}+h_{i-1}\delta_{i-1}\beta_i)^2}\left(\frac{b_{i-1}}{\sigma_{i-1}} + \frac{h_{i-1}\delta_{i-1}}{\sigma_{i-1}(1-\exp(-\sigma_{i-1}))} - \frac{h_{i-1}\delta_{i-1}}{\sigma_{i-1}^2}\right)\right| \le c_{40}\varepsilon$$

with a constant c_{40} independent of ε and h. Thus, for $\varepsilon \ll 1$ the right-hand side of (1.100) is a contraction operator on $[0,1]$ with a sufficiently small contraction coefficient of order ε. Therefore we define β_i as the limit of the iterative process

$$\beta_i = \lim_{j\to\infty} s_j$$

where

$$\begin{aligned} s_0 &= \beta_{const,i}, \\ s_{j+1} &= \frac{\varepsilon}{h_i(b_{i-1}+s_j(b_i-b_{i-1}))} \\ &\quad - \frac{\exp(-(b_{i-1}+s_j(b_i-b_{i-1}))h_i/\varepsilon)}{1-\exp(-(b_{i-1}+s_j(b_i-b_{i-1}))h_i/\varepsilon)}. \end{aligned} \tag{1.101}$$

Then $\alpha_i = 1 - \beta_i$ is determined from (1.74).

The numerical experiments confirm the fast of convergence of the iterative process (1.101). When calculations were performed for the model problem, 2-4 iterations were need to obtain an accuracy of 10^{-7}.

2 Two-dimensional convection-diffusion problem

2.1 General remarks

2.1.1 Qualitative behaviour of the solution

Let Ω be the unit square $(0,1) \times (0,1)$ with boundary Γ. Consider the Dirichlet problem

$$Lu \equiv -\varepsilon\, \Delta u + \frac{\partial}{\partial x}(b(x)u) = f \qquad \text{in} \quad \Omega, \tag{2.1}$$

$$u = 0 \qquad \text{on} \quad \Gamma. \tag{2.2}$$

Here, as usually, $\varepsilon \ll 1$ is a positive small parameter. The functions $b(x)$ and $f(x,y)$ are sufficiently smooth:

$$b \in C^3([0,1]), \quad f(x,y) \in C^2(\bar{\Omega}). \tag{2.3}$$

Under this assumptions the problem (2.1), (2.2) has a unique solution in $C^2(\Omega)$ (see, e.g., [87]).

The behaviour of the solution in the two-dimensional case is more complicated than in the one-dimensional case. In addition to the exponentional (regular) boundary layer, as in Chapter 1, there is a parabolic boundary layer that arises near some parts of the boundary. The boundary layer of this type is formed due to the fact that the characteristics of the reduced problem (for $\varepsilon = 0$) is tangent to the boundary. Besides, corner boundary layers can arise at the vertices of square.

Let the conditions

$$0 < B_1 \le b(x) \le B_2 < \infty, \quad x \in [0,1]; \tag{2.4}$$

$$f(0,0) = f(1,0) = f(0,1) = f(1,1) = 0 \tag{2.5}$$

be fulfilled. Then the solution of the problem (2.1) – (2.2) belongs to $C^3(\bar{\Omega})$ ([40]). Notice that the derivatives up to the third order are continuous and hence are bounded on $\bar{\Omega}$. But the constants in the estimates of this derivatives depend on ε and increase indefinitely as ε tends to zero.

Let us introduce the notations

$$\Gamma_{in} = \{(x,y) : x = 0, \quad y \in [0,1]\},$$
$$\Gamma_{out} = \{(x,y) : x = 1, \quad y \in [0,1]\},$$
$$\Gamma_{tg} = \{(x,y) : x \in [0,1], \quad y = 0,1\}.$$

Here the regular boundary layer arises near Γ_{out} and the parabolic boundary layer arises along Γ_{tg} (see Fig. 2).

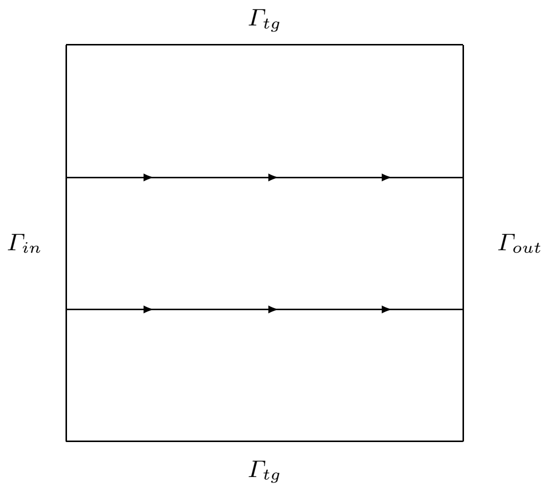

Fig. 2: Domain Ω.

Notice that in general the operator corresponding to the left-hand side of (2.1) with the mixed boundary conditions does not satisfy the maximum principle (for example, for $b' < 0$), however the comparison principle still holds. Later the comparison principle is applied to the differential operator of the form

$$\mathcal{L}u \equiv -\varepsilon \Delta u + b\frac{\partial u}{\partial x} + du \tag{2.6}$$

where $b(x)$ satisfies the assumptions (2.3), (2.4) and $d(x)$ is a bounded function on $[0, 1]$ that is defined in each individual case.

Lemma 13. *Let* $\varepsilon > 0$ *be small enough. Assume that* (2.3), (2.4) *hold and* $u, w \in C^2(\Omega) \bigcap C(\bar{\Omega})$ *satisfy*

$$|\mathcal{L}u| \leq \mathcal{L}w \quad in \quad \Omega, \qquad |u| \leq w \quad on \quad \Gamma. \tag{2.7}$$

Then the estimate

$$|u| \leq w \qquad on \quad \bar{\Omega} \tag{2.8}$$

is valid.

Proof. Introduce the functions

$$v(x, y) = u(x, y)\exp(-\sigma x) \qquad \text{and} \qquad z(x, y) = w(x, y)\exp(-\sigma x) \tag{2.9}$$

with the constant

$$\sigma = 1 + \max_{x\in[0,1]} \frac{|d(x)|}{b(x)}. \tag{2.10}$$

Transform the differential operator $\mathcal{L}$ into

$$\left(\widetilde{\mathcal{L}}\Phi(x,y)\right) \equiv \exp(-\sigma x)\mathcal{L}\left(\Phi(x,y)\exp(\sigma x)\right), \tag{2.11}$$

that gives

$$\widetilde{\mathcal{L}}\Phi(x,y) = -\varepsilon\,\Delta\Phi(x,y) + (b(x) - 2\varepsilon\sigma)\frac{\partial\Phi(x,y)}{\partial x} + (d(x) + \sigma b(x) - \varepsilon\sigma^2)\Phi(x,y).$$

Assume that

$$\varepsilon \in \Big(0, B_1/(4\sigma^2)\Big]. \tag{2.12}$$

Taking into consideration the definition of σ and the smallness of ε, we obtain

$$\begin{aligned} d + \sigma b - \varepsilon\sigma^2 &= -|d| + b + |d| - B_1/2 \geq B_1/2 \geq 0 \quad \text{on} \quad [0,1], \\ b - 2\varepsilon\sigma &\geq b - 2B_1\sigma/4\sigma^2 = b - B_1/2\sigma \geq B_1/2 \quad \text{on} \quad [0,1]. \end{aligned} \tag{2.13}$$

From (2.7) we have

$$\begin{aligned} &|\widetilde{\mathcal{L}}v| \leq \widetilde{\mathcal{L}}z \qquad \text{in} \quad \Omega, \\ &|v| \leq z \qquad \text{on} \quad \Gamma. \end{aligned}$$

The operator $\widetilde{\mathcal{L}}$ satisfies the maximum principle (see [114]). As a consequence we obtain

$$|v| \leq z \qquad \text{on} \quad \bar{\Omega}.$$

Multiplying the last inequality by $\exp(\sigma x)$, we get (2.8). □

When on Γ_{tg} we can estimate not a function u but its normal derivative only, the comparison principle also holds.

Lemma 14. *Let $\varepsilon > 0$ be small enough. Assume that* (2.3), (2.4) *hold, and $u, w \in C^2(\bar{\Omega})$ satisfy*

$$\begin{aligned} &|\mathcal{L}u| \leq \mathcal{L}w \qquad \textit{in} \quad \Omega, \\ &|u| \leq w \qquad \textit{on} \quad \Gamma \setminus \Gamma_{tg}, \qquad \left|\frac{\partial u}{\partial n}\right| \leq \frac{\partial w}{\partial n} \quad \textit{on} \quad \Gamma_{tg}. \end{aligned} \tag{2.14}$$

Then the estimate

$$|u| \leq w \qquad \textit{on} \quad \bar{\Omega} \tag{2.15}$$

be valid.

Proof. We introduce the constant σ by (2.10) and assume that ε satisfies (2.12). We use (2.9) and consider the operator $\tilde{\mathcal{L}}$ from (2.11). Then $\tilde{\mathcal{L}}$ satisfies the maximum principle again. Notice that on Γ_{tg} we have $\partial/\partial n = \pm\partial/\partial y$, therefore

$$|\widetilde{\mathcal{L}}v| \le \widetilde{\mathcal{L}}z \qquad \text{in} \quad \Omega,$$
$$|v| \le z \qquad \text{on} \quad \Gamma_{in} \cup \Gamma_{out}, \qquad \left|\frac{\partial v}{\partial n}\right| \le \frac{\partial z}{\partial n} \quad \text{on} \quad \Gamma_{tg}.$$

First we prove (by contradiction) the statement of the lemma in the case of $z = 0$ and, consequently, $v = 0$ on $\bar{\Omega}$. For this purpose we suppose that there exists a point $(x, y) \in \bar{\Omega}$ where $v(x, y) < 0$. Assume that at a point $(x_0, y_0) \in \bar{\Omega}$ we have

$$v(x_0, y_0) = \min_{\bar{\Omega}} v(x, y) < 0. \tag{2.16}$$

Since $\widetilde{\mathcal{L}}$ satisfies the maximum principle, (x_0, y_0) does not belong to Ω. Because of the condition on $\Gamma_{in} \cup \Gamma_{out}$ the point (x_0, y_0) does not belong to this part of the boundary. It remains that $(x_0, y_0) \in \Gamma_{tg}$. Assume that, for definiteness, $y_0 = 1$. Because of the condition on Γ_{tg} we have

$$\frac{\partial v}{\partial n}(x_0, 1) = \frac{\partial v}{\partial y}(x_0, 1) \ge 0.$$

If $\partial v/\partial y(x_0, 1) > 0$ then due to continuity there exists an interval $[1 - \delta, 1]$ in y on which this inequality holds. Use the Taylor expansion

$$v(x_0, 1 - \delta) = v(x_0, 1) - \delta\frac{\partial v}{\partial y}(x_0, \eta), \quad \eta \in [1 - \delta, 1].$$

It implies $v(x_0, 1 - \delta) < v(x_0, 1)$ that is in contradiction with (2.16). Therefore

$$\frac{\partial v}{\partial y}(x_0, 1) = 0.$$

Applying this reasoning to the second derivative, we obtain

$$\frac{\partial^2 v}{\partial y^2}(x_0, 1) \ge 0.$$

In a similar way we get

$$\frac{\partial v}{\partial x}(x_0, 1) = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2}(x_0, 1) \ge 0.$$

Using the above four relations in the expression $(\widetilde{\mathcal{L}}v)(x_0, 1)$, we obtain

$$(\widetilde{\mathcal{L}}v)(x_0, 1) \le v(x_0, 1)B_1/2 < 0.$$

This is in contradiction with the condition $\widetilde{\mathcal{L}}v \ge 0$ on $\bar{\Omega}$ which follows from the same condition on Ω and from the continuity of v and its first and second derivatives. Thus, our assumption that v can take negative values is wrong. Hence, $v \ge 0$ on $\bar{\Omega}$.

Finally, using the last statement for the functions $z - v$ and $z + v$, we obtain $z - v \ge 0$, $z + v \ge 0$. Hence $|v| \le z$ on $\bar{\Omega}$ that implies

$$|u| \le w \quad \text{on} \quad \bar{\Omega}. \qquad \Box$$

2.1.2 The weak formulation

Multiply (2.1) by an arbitrary function $v \in \overset{\circ}{W}{}_2^1(\Omega)$. By applying Green's formula we obtain the weak formulation: *find* $u \in \overset{\circ}{W}{}_2^1(\Omega)$ *such that for all* $v \in \overset{\circ}{W}{}_2^1(\Omega)$

$$a(u, v) = (f, v) \tag{2.17}$$

with the bilinear form

$$a(u, v) = \int_\Omega \left(\varepsilon \, \nabla u \nabla v - bu \frac{\partial v}{\partial x} \right) \, d\Omega \tag{2.18}$$

and the inner product

$$(f, v) = \int_\Omega f v \, d\Omega. \tag{2.19}$$

Let us introduce the norm

$$\|v\|_\infty = \sup_{\bar{\Omega}} \operatorname{vrai} |v|.$$

We use the notations ∂_1 for a partial derivative $\partial/\partial x$ and ∂_2 for a partial derivative $\partial/\partial y$. Similarly we denote the second derivatives by $\partial_{22} = \partial_2(\partial_2)$ and so on.

2.2 The scheme with the fitted quadrature rule for a problem without parabolic boundary layers

In this section we consider the method for the problem (2.1) – (2.2) with a solution free of a parabolic boundary layer near Γ_{tg}.

2.2.1 The differential problem

Assume that

$$f(x, y) = 0 \qquad \text{on} \quad \Gamma_{tg}. \tag{2.20}$$

Then under conditions (2.4), (2.20) the first and second partial derivatives of the solution of the problem (2.1) – (2.2) with respect to y are bounded. Namely, the following estimates hold.

Lemma 15. *Assume that* $0 < \varepsilon \ll 1$ *and* (2.3), (2.4), (2.20) *are valid for the problem* (2.1)–(2.2). *Then we have*

$$\|u\|_\infty + \|\partial_2 u\|_\infty + \|\partial_{22} u\|_\infty \le c_1. \tag{2.21}$$

Proof. Assume that $d = b'(x)$ and σ is given by (2.10). Take the barrier function

$$w(x, y) = c_2 \exp(\sigma x) \quad \text{where} \quad c_2 = 2\|f\|_\infty / B_1.$$

Taking into consideration (2.13), (2.2), and (2.4) we have

$$\mathcal{L}w(x, y) \ge \|f\|_\infty \ge |Lu(x, y)| \quad \text{in} \quad \Omega,$$

$$w(x, y) \ge |u(x, y)| \quad \text{on} \quad \Gamma.$$

Thus, applying Lemma 13 and using the upper bound of the function w, we conclude that the solution u is bounded uniformly with respect to ε.

To prove the estimate for the first derivative on Ω, we differentiate the equation (2.1) with respect to y and introduce the notation $v_1 = \partial_2 u$. Then we get

$$\mathcal{L}v_1 = \partial_2 f \quad \text{in} \quad \Omega.$$

Since $u(0, y) = u(1, y) = 0$, we obtain

$$v_1 = 0 \quad \text{on} \quad \Gamma \setminus \Gamma_{tg}. \tag{2.22}$$

From (2.1), (2.2), (2.20) we have

$$\partial_2 v_1 = \partial_{22} u = 0 \quad \text{on} \quad \Gamma_{tg}. \tag{2.23}$$

Now, setting $c_2 = 2\|\partial_2 f\|_\infty / B_1$ and taking into account (2.13), (2.4), (2.22), and (2.23), we see that the barrier function $w(x, y) = c_2 \exp(\sigma x)$ satisfies the relations

$$|\mathcal{L}v_1| \le \mathcal{L}w \quad \text{in} \quad \Omega,$$

$$|v_1| \le w \quad \text{on} \quad \Gamma \setminus \Gamma_{tg}, \qquad \left|\frac{\partial v_1}{\partial n}\right| \le \frac{\partial w}{\partial n} \quad \text{on} \quad \Gamma_{tg}.$$

Thus, by Lemma 14 v_1 is bounded on $\bar{\Omega}$ by the function w which satisfies the estimate $w \leq c_2 \exp(\sigma)$ on $\bar{\Omega}$. Hence, $\partial_2 u$ is uniformly bounded on $\bar{\Omega}$ by a constant independent of ε.

It remains to show that the second derivative $\partial_{22} u$ is bounded. To do this, we twice differentiate (2.1) with respect to y, put $v_2 = \partial_{22} u$, and to use Lemma 13 with the same barrier function $w(x, y)$ and with the constant $c_2 = 2\|\partial_{22} f\|_\infty / B_1$. $\square$

Let us consider the following expansion of the solution

$$u = v_0 + \rho_0 + \varepsilon\eta. \tag{2.24}$$

Here v_0 is the solution of the reduced problem

$$\partial_1 (b(x) v_0) = f(x, y) \quad \text{in} \quad \Omega, \tag{2.25}$$

$$v_0 = 0 \quad \text{on} \quad \Gamma_{in}. \tag{2.26}$$

The function ρ_0 is the regular boundary layer component

$$\rho_0(x, y) = g(y) s(x) \exp\left(-(1 - x) b(x)/\varepsilon\right) \tag{2.27}$$

where $g(y) = -v_0(1, y)$ and $s(t)$ is the cut-off function $s \in C^3([0, 1])$ satisfying (1.9). The solution of the problem (2.25), (2.26) has the form

$$v_0(x, y) = \frac{1}{b(x)} \int_0^x f(t, y)\, dt. \tag{2.28}$$

Due to (2.3) we have $v_0 \in C^2(\bar{\Omega})$. Because of (2.20)

$$v_0(x, y) = 0 \quad \text{on} \quad \Gamma_{tg}.$$

In view of the definitions of ρ_0, g, s and with (2.20) we get

$$\rho_0(x, y) = 0 \quad \text{on} \quad \Gamma \setminus \Gamma_{out}, \quad \rho_0(1, y) = -v_0(1, y) \quad \text{on} \quad \Gamma_{out}. \tag{2.29}$$

To estimate the remainder term in the expansion (2.24) we need the following lemma.

Lemma 16. *Assume that* $\varepsilon > 0$ *is small enough and* (2.4) *hold. Let* $s(x) \in C^3([0, 1])$ *be the cut-off function* (1.9). *Then the function*

$$\psi_1(x) = \frac{1 - x}{\varepsilon} \exp(-(1 - x) b(x)/\varkappa\varepsilon) s(x) \tag{2.30}$$

with a fixed constant $\varkappa \in (1, 2)$ *satisfies the inequality*

$$\mathcal{L}\psi_1 \geq c_1 \left(\frac{1}{\varepsilon} + \frac{1 - x}{\varepsilon^2}\right) \exp(-(1 - x) b(x)/\varkappa\varepsilon) - c_2 \tag{2.31}$$

for the operator $\mathcal{L}$ *from* (2.6) *with the function* $d = b'(x)$ *and some positive constants* c_1 *and* c_2 *independent of* ε.

Proof. We introduce the notation

$$E_1(x,\varepsilon) = \exp\left(-\frac{(1-x)b(x)}{\varkappa\varepsilon}\right).$$

Then we obtain the relation

$$\begin{aligned}\mathcal{L}\psi_1 &= \frac{1}{\varepsilon}b(x)\left(\frac{2}{\varkappa}-1\right)s(x)E_1(x,\varepsilon) + \frac{1-x}{\varepsilon^2}b^2(x)\left(\frac{1}{\varkappa}-\frac{1}{\varkappa^2}\right)s(x)E_1(x,\varepsilon)\\ &+ \frac{1-x}{\varepsilon}a_1(x)s(x)E_1(x,\varepsilon) + \left(\frac{1-x}{\varepsilon}\right)^2 a_2(x)s(x)E_1(x,\varepsilon)\\ &+ \left(2+\frac{1-x}{\varepsilon}\right)a_3(x)s'(x)E_1(x,\varepsilon) + (1-x)s''(x)E_1(x,\varepsilon)\end{aligned} \tag{2.32}$$

with bounded functions a_1, a_2, a_3. First we consider the right-hand side of (2.32) on the segment $[2/3,1]$. Remember that $s=1$ and $s'=s''=0$ on this segment. Due to the definition of $\varkappa$ the coefficients $2/\varkappa-1$ and $1/\varkappa-1/\varkappa^2$ are positive. Since $b(x)\ge B_1>0$, the sum of the first and second terms in the right-hand side of (2.32) has the lower bound

$$c_3\left(\frac{1}{\varepsilon}+\frac{1-x}{\varepsilon^2}\right)E_1(x,\varepsilon).$$

To estimate the remaining two nonzero terms, we use the inequality

$$x^\alpha\exp(-\beta x)\le(\alpha/\beta)^\alpha\exp(-\alpha),\quad x\in[0,\infty) \tag{2.33}$$

which holds for each $\alpha\ge 0,\ \ \beta>0$ (see [4]). Setting $t=(1-x)/\varepsilon$ and $t=(1-x)^2/\varepsilon^2$, and using the fact that a_1 and a_2 are bounded, we estimate these two terms from below by a negative constant $-c_4$. Hence we have

$$\mathcal{L}\psi_1\ge c_3\left(\frac{1}{\varepsilon}+\frac{1-x}{\varepsilon^2}\right)E_1(x,\varepsilon)-c_4\quad\text{on}\quad[2/3,1]. \tag{2.34}$$

Now we consider (2.32) on the segment $x\in[0,2/3]$. The right-hand side can be expressed as

$$\mathcal{L}\psi_1=\left(a_4(x)+\frac{1}{\varepsilon}a_5(x)+\frac{1}{\varepsilon^2}a_6(x)\right)E_1(x,\varepsilon)$$

where the functions a_4,a_5,a_6 are bounded on $[0,2/3]\times[0,1]$. Ones, we use (2.33) for $t=1/\varepsilon$ and $\alpha=0,1,2$. This gives

$$\mathcal{L}\psi_1\ge -c_5\quad\text{on}\quad[0,2/3]. \tag{2.35}$$

In a similar way for the expression in the right-hand side of (2.31) we obtain the upper bound

$$\left(\frac{1}{\varepsilon} + \frac{1-x}{\varepsilon^2}\right) E_1(x,\varepsilon) \le c_6 \quad \text{on} \quad [0, 2/3]. \tag{2.36}$$

Let us set

$$c_1 = c_3 \quad \text{and} \quad c_2 = \max\{c_4, c_3 c_6 + c_5\}. \tag{2.37}$$

Thus, the estimate (2.34) involves (2.31) on the segment $[2/3, 1]$. On the remaining segment $[0, 2/3]$ from (2.35)–(2.37) we get

$$\mathcal{L}\psi_1 \ge -c_5 \ge c_3 c_6 - c_2 \ge c_1 \left(\frac{1}{\varepsilon} + \frac{1-x}{\varepsilon^2}\right) E_1(x,\varepsilon) - c_2.$$

This estimate together with (2.34) completes the proof. □

Lemma 17. *Assume that* $\varepsilon > 0$ *is small enough and* (2.4) *hold. Then the function*

$$\psi_2(x) = (1 - \exp(-(1-x)B_2/\varepsilon))\,(\exp(\sigma x) - 1) \tag{2.38}$$

with the constant σ *from* (2.10) *satisfies the inequality*

$$(\mathcal{L}\psi_2)\,(x,y) \ge \frac{1}{4} B_1 \exp \sigma x \quad \text{on} \quad \Omega \tag{2.39}$$

where the operator $\mathcal{L}$ *is given by* (2.6) *with* $d = b'(x)$.

Proof. Introduce the notation

$$E_2(x,\varepsilon) = \exp\left(-\frac{(1-x)B_2}{\varepsilon}\right).$$

Then we obtain the relation

$$\begin{aligned}
\mathcal{L}\psi_2 = {} & \frac{B_2^2}{\varepsilon} E_2(x,\varepsilon)\,(\exp(\sigma x) - 1) + 2\sigma B_2 E_2(x,\varepsilon) \exp(\sigma x) \\
& - \varepsilon\sigma^2 \,(1 - E_2(x,\varepsilon)))\exp(\sigma x) - b\frac{B_2}{\varepsilon} E_2(x,\varepsilon)(\exp(\sigma x) - 1) \\
& + b\sigma\,(1 - E_2(x,\varepsilon))\exp(\sigma x) + b'\,(1 - E_2(x,\varepsilon))\,(\exp(\sigma x) - 1).
\end{aligned}$$

Due to the upper estimate of $b(x)$ in (2.4) the sum of the first and fourth terms is nonnegative. We discard it and use simple transformations:

$$\begin{aligned}
\mathcal{L}\psi_2 \ge {} & (-\varepsilon\sigma^2 + b\sigma - |b'|)\exp(\sigma x) \\
& + (2\sigma B_2 + \varepsilon\sigma^2 - b\sigma - |b'|)\exp(-(1-x)B_2/\varepsilon)\exp(\sigma x).
\end{aligned}$$

For $\varepsilon \le B_1/(2\sigma^2)$ in view of (2.10) the inequality

$$|b'(x)| \le b(x)\sigma/2.$$

Then we have

$$-\varepsilon\sigma^2+b(x)\sigma-|b'(x)| \ge \frac{1}{4}B_1, \quad 2\sigma B_2+\varepsilon\sigma^2-b(x)\sigma-|b'(x)| \ge \varepsilon\sigma^2+\frac{1}{2}B_1 \ge 0.$$

Hence we obtain

$$\mathcal{L}\psi_2 \ge \frac{1}{4}B_1 \exp(\sigma x).$$

That completes the proof of the lemma.□

Lemma 18. *Let $\varepsilon > 0$ be small enough and the operator $\mathcal{L}$ be defined by* (2.6) *with a bounded function $d(x)$. Then the function*

$$\psi_3(x) = \left(1 + \frac{1}{\varepsilon}\exp(-(1-x)B_1/2\varepsilon)\right)\exp(\sigma x) \tag{2.40}$$

with the constant σ from (2.10) *satisfies the inequality*

$$\begin{aligned} \mathcal{L}\psi_3 \ge \frac{B_1^2}{8\varepsilon^2}&\exp(-(1-x)B_1/2\varepsilon)\exp\sigma x \\ +\frac{B_1}{2}&\left(1+\exp(-(1-x)B_1/2\varepsilon)\right)\exp\sigma x. \end{aligned} \tag{2.41}$$

Proof. Introduce the notation

$$E_3(x,\varepsilon) = \exp\left(-\frac{(1-x)B_1}{2\varepsilon}\right)$$

and assume that

$$\varepsilon \le \min\left\{\frac{B_1}{2\sigma^2}, \frac{B_1}{8\sigma}\right\}. \tag{2.42}$$

Then we obtain the relation

$$\begin{aligned} (\mathcal{L}\psi_3)(x,y) = &\left(\frac{B_1 b_1}{2\varepsilon^2} - \frac{B_1^2}{4\varepsilon^2} - \frac{B_1\sigma}{\varepsilon}\right)E_3(x,\varepsilon)\exp(\sigma x) \\ &+ \left(-\varepsilon\sigma^2 + b_1\sigma + d\right)\left(1+\varepsilon^{-1}E_3(x,\varepsilon)\right)\exp(\sigma x). \end{aligned}$$

Because of (2.4) and (2.42) the factor in the first term is estimated from below:

$$\frac{B_1 b_1}{2\varepsilon^2} - \frac{B_1^2}{4\varepsilon^2} - \frac{B_1\sigma}{\varepsilon} \ge \frac{B_1^2}{4\varepsilon^2} - \frac{B_1\sigma}{\varepsilon} \ge \frac{B_1^2}{8\varepsilon^2}.$$

The factor in the second term is evaluated from below due to (2.13):

$$-\varepsilon\sigma^2 + b_1\sigma + d \geq B_1/2.$$

These three inequalities involve (2.41). □

The following Lemma describes the behaviour of the remainder term in the expansion (2.24) and of its derivatives.

Lemma 19. *Let* $\varepsilon > 0$ *be small enough and* (2.3), (2.4), (2.20) *be valid for the problem* (2.1)–(2.2). *Then the remainder term* η *in* (2.24) *satisfies the estimates*

$$\|\eta\|_\infty \leq c_7, \tag{2.43}$$

$$|\partial_1\eta(x,y)| \leq c_8(1 + \varepsilon^{-1}\exp(-B_1(1-x)/2\varepsilon)), \quad (x,y) \in \bar{\Omega}, \tag{2.44}$$

$$\|\partial_{22}\eta\|_\infty \leq c_9\varepsilon^{-1}. \tag{2.45}$$

Proof. First we set $d = b'(x)$ in (2.6). Then we get $\mathcal{L}u = f$. Simple calculations show that η in (2.24) satisfies

$$L\eta = \tilde{f} \equiv a_0(x,y) + \frac{1}{\varepsilon}a_1(x,y)A(x,\varepsilon) + \frac{1-x}{\varepsilon^2}a_2(x,y)A(x,\varepsilon) \text{ on } \bar{\Omega} \tag{2.46}$$

where

$$A(x) = \exp(-(1-x)b(x)/\varepsilon)$$

and a_0, a_1, a_2 are bounded functions on $\bar{\Omega}$. Therefore the right-hand side of (2.46) is estimated in the following way:

$$|L\eta| \leq c_{10} + \left(c_{11}\frac{1}{\varepsilon} + c_{12}\frac{1-x}{\varepsilon^2}\right)A(x,\varepsilon) \tag{2.47}$$

with appropriate constants c_{10}, c_{11}, and c_{12}. Let us use the barrier function

$$w(x,y) = c_{13}\psi_1(x,y) + c_{14}\psi_2(x,y)$$

with constants

$$c_{13} = \max\{c_{11}, c_{12}\}/c_1 \quad \text{and} \quad c_{14} = 4(c_{10} + c_2c_3)/B_1$$

where the functions ψ_1, ψ_2 are given in Lemmata 16 and 17. This yields

$$|(\mathcal{L}\eta)(x,y)| \leq (\mathcal{L}w)(x,y) \quad \text{in } \Omega.$$

Moreover, we have

$$w \geq 0 = |\eta| \quad \text{on} \quad \Gamma.$$

Thus, all the assumptions of Lemma 16 are satisfied and consequently (2.43) holds. Together with the estimates for the functions $\psi_1,\ \psi_2$ this implies the inequality

$$|\eta(x,y)| \le w(x,y). \tag{2.48}$$

Moreover, since

$$w(0,y) = w(1,y) = 0 \quad \forall\, y \in\ [0,1],$$

from (2.48) we have

$$|\partial_1 \eta(0,y)| \le c_{15} \quad \text{and} \quad |\partial_1 \eta(1,y)| \le c_{16}\varepsilon^{-1}. \tag{2.49}$$

In order to prove (2.44) we differentiate (2.46) with respect to x. Introduce the notation $\zeta = \partial_1 \eta$ and set $d = 2b'$ in (2.6). Then we obtain

$$\begin{aligned}(\mathcal{L}\zeta)(x,y) = {} & a_3(x,y) \\ & + \frac{1}{\varepsilon^2} a_4(x,y)A(x,\varepsilon) + \frac{1-x}{\varepsilon^3} a_5(x,y)A(x,\varepsilon)\end{aligned} \tag{2.50}$$

where the functions a_3, a_4, and a_5 are bounded on $\bar{\Omega}$. The right-hand side of (2.50) is estimated in the following way:

$$|(\mathcal{L}\zeta)(x,y)| \le c_{17} + c_{18}\varepsilon^{-2} A(x,\varepsilon). \tag{2.51}$$

Now we take the barrier function $w(x,y) = c_{19}\psi_3(x,y)$ from Lemma 18 with the constant $c_{19} = \max\{8c_{17}/B_1^2,\ 2c_{19}/B_1\}$. We get

$$|(\mathcal{L}\zeta)(x,y)| \le (\mathcal{L}w)(x,y) \quad \text{in } \Omega,$$

$$w \ge 0 = |\zeta| \quad \text{on} \quad \Gamma_{tg}, \quad w \ge |\zeta| \quad \text{on} \quad \Gamma_{in} \cup \Gamma_{out}.$$

The last inequality follows from (2.49). Thus, due to Lemma 13 we obtain

$$|\zeta| \le w \quad \text{on} \quad \bar{\Omega}$$

that involves (2.44).

In order to prove (2.45) we consider the equality

$$\partial_{22}\eta = \frac{1}{\varepsilon}(\partial_{22}u - \partial_{22}v_0 - \partial_{22}\rho_0) \tag{2.52}$$

which follows from (2.24). Taking into consideration (2.20), (2.27), and (2.28), we obtain (2.45). □

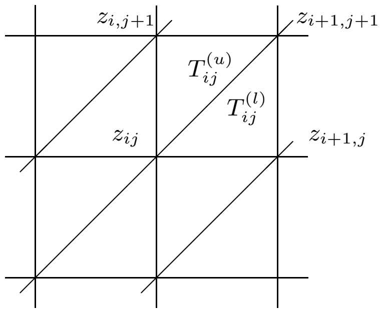

Fig. 3: The fragment of the triangulation $\mathcal{T}_h$.

2.2.2 Construction of the quadrature rule

For the implementation of the Galerkin method we construct an uniform triangulation $\mathcal{T}_h$. To do this, we consider the grid

$$x_i = ih, \qquad y_j = jh, \qquad i, j = 0, 1, \ldots, n,$$

with the mesh size $h = 1/n$ for integer $n \geq 2$. We denote the set of nodes by

$$\bar{\Omega}_h = \{z_{ij} = (x_i, y_j),\ i, j = 0, 1, \ldots, n\},$$

the set of interior nodes by

$$\Omega_h = \{z_{ij} = (x_i, y_j),\ i, j = 1, 2, \ldots, n-1\},$$

and the set of boundary nodes by

$$\Gamma_h = \{z_{ij} = (x_i, y_j),\ i = 0, 1,\ j = 0, 1, \ldots, n;\ i = 0, 1, \ldots, n,\ j = 0, 1\}.$$

Then the triangulation $\mathcal{T}_h$ is constructed by dividing each elementary rectangle $\Omega_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ into two *elementary* triangles by the diagonal passing from (x_i, y_j) to (x_{i+1}, y_{j+1}) (see Fig. 3).

At each node $z_{ij} \in \Omega_h$ we introduce the basis function φ_{ij} which equals 1 at the node z_{ij}, equals 0 at any other node of $\bar{\Omega}_h$, and is linear on each elementary triangle of $\mathcal{T}_h$. Denote the linear span of these functions by

$$H^h = \operatorname{span}\{\varphi_{ij}\}_{i,j=1}^{n-1}.$$

With these notations, we arrive at the Galerkin problem: *find* $u^h \in H^h$ *such that*

$$a(u^h, v^h) = (f, v^h) \qquad \forall\, v^h \in H^h. \tag{2.53}$$

But the solution of this problem is unstable and has poor accuracy because of the boundary layer component ([9]). In the same way as in one-dimensional case, we provide the stability and improve the accuracy by the special approximation of the bilinear form a with the fitted quadrature rule.

Let $T_{ij}^{(l)}$ (or $T_{ij}^{(u)}$, respectively) be an arbitrary triangle of $\mathcal{T}_h$ with the vertices $z_{i,j}$, $z_{i+1,j+1} = (x_{i+1}, y_{j+1})$, and $z_{i+1,j} = (x_{i+1}, y_j)$ ($z_{i+1,j} = (x_i, y_{j+1})$, respectively) as in Fig. 3. We denote the elementary part of the bilinear form (2.18) on an arbitrary triangle $T = T_{ij}^{(l)}$ or $T = T_{ij}^{(u)}$ by

$$a_T(u,v) = \int_T \left((\varepsilon \partial_1 u - bu)\, \partial_1 v + \varepsilon \partial_2 u \partial_2 v \right) d\Omega. \tag{2.54}$$

In principle, freezing the coefficient b on a triangle T is enough to perform the integration exactly. But the accuracy of this formula is unsatisfactory because of the boundary layer function ρ_0. Therefore, we try to get another quadrature rule.

Thus, we apply the three-point quadrature rule on a triangle $T = T_{ij}^{(l)}$ for the approximation of the bilinear form (2.54):

$$\int_T g(x,y) d\Omega \approx \frac{h^2}{2} \left(\alpha_{1i} g(z_{ij}) + \alpha_{2i} g(z_{i+1,j}) + \alpha_{3i} g(z_{i+1,j+1}) \right).$$

Then an elementary contribution of the algebraic bilinear form can be expressed as

$$\begin{aligned} a^h_{T^{(l)}_{ij}}(w^h, v^h) = \frac{h^2}{2} \Big(\big(\varepsilon \partial_1 w^h - b_i (\alpha_{1i} w^h(z_{i,j}) + \alpha_{2i} w^h(z_{i+1,j}) \\ + \alpha_{3i} w^h(z_{i+1,j+1})) \big) \partial_1 v^h + \varepsilon \partial_2 w^h \partial_2 v^h \Big). \end{aligned} \tag{2.55}$$

From here on we use the notation $b_i = b(x_i)$. We choose the weights $\alpha_{k,i}$ from the following two requirements. Firstly, in order to guarantee the first order accuracy for smooth functions, the quadrature rule have to be exact for constant functions. This immediately gives the equation

$$\alpha_{1i} + \alpha_{2i} + \alpha_{3i} = 1. \tag{2.56}$$

Secondly, we try to minimize the difference

$$a_T(\rho_0, v^h) - a^h_T(\rho^I_0, v^h), \qquad v^h \in H^h, \tag{2.57}$$

for the regular boundary layer function ρ_0 and its piecewise linear interpolant $\rho_0^I \in H^h$. For this purpose we put

$$\begin{aligned} &\int_T (\varepsilon \partial_x \zeta_i - b_i \zeta_i)\, \partial_1 v^h d\Omega \\ &= \frac{h^2}{2}\Big(\varepsilon \partial_1 \zeta_i^I - b_i\left(\alpha_{1i}\zeta_i^I(z_{i,j}) + \alpha_{2i}\zeta_i^I(z_{i+1,j})\alpha_{3i}\zeta_i^I(z_{i+1,j+1})\right)\Big)\partial_1 v^h \end{aligned} \tag{2.58}$$

for the function

$$\zeta_i(x) = \exp\left(-(1-x)b_i/\varepsilon\right)$$

and its piecewise linear interpolant $\zeta_i^I(x,y)$ on $T_{ij}^{(l)}$.

To diminish the difference stencil, we put $\alpha_{3i} = 0$. Thus, for the parameters of the quadrature rule we have the system of linear algebraic equations

$$\begin{aligned} \alpha_{1i} + \exp(\sigma_i)\alpha_{2i} + \exp(\sigma_i)\alpha_{3i} &= \frac{1}{\sigma_i}\left(\exp\sigma_i - 1\right), \\ \alpha_{1i} + \alpha_{2i} + \alpha_{3i} &= 1, \\ \alpha_{3i} &= 0 \end{aligned} \tag{2.59}$$

where $\sigma_i = b_i h/\varepsilon$. It has the unique solution

$$\alpha_{1i} = \frac{\exp\sigma_i}{(\exp\sigma_i - 1)} - \frac{1}{\sigma_i}, \quad \alpha_{2i} = \frac{1}{\sigma_i} - \frac{1}{\exp\sigma_i - 1}, \quad \alpha_{3i} = 0. \tag{2.60}$$

With the weights obtained we rewrite (2.55) in the following form:

$$\begin{aligned} a^h_{T_{ij}^{(l)}}(w^h, v^h) = \frac{h^2}{2}\Big(&\frac{b_i}{\exp\sigma_i - 1}(w^h_{i+1,j} - w^h_{ij}\exp\sigma_i)\partial_1 v^h \\ &+ \varepsilon \partial_2 w^h \partial_2 v^h\Big). \end{aligned} \tag{2.61}$$

From here on we use the notation $v_{ij} = v(z_{ij})$ for any function $v(x,y)$.

In a similar way on the triangle $T_{ij}^{(u)}$ we obtain the following approximation of the bilinear form (2.54):

$$\begin{aligned} a^h_{T_{ij}^{(u)}}(w^h, v^h) = \frac{h^2}{2}\Big(&\frac{b_i}{\exp\sigma_i - 1}(w^h_{i+1,j+1} - w^h_{i,j+1}\exp\sigma_i)\frac{\partial v^h}{\partial x} \\ &+ \varepsilon \frac{\partial w^h}{\partial y}\frac{\partial v^h}{\partial y}\Big). \end{aligned} \tag{2.62}$$

To integrate the right-hand side, we use the simple quadrature rules

$$\int_{T_{ij}^{(l)}} fv\, d\Omega \approx \frac{1}{6}h^2(f_{ij}v_{ij} + f_{i+1,j}v_{i+1,j} + f_{i+1,j+1}v_{i+1,j+1}),$$

$$\int_{T_{ij}^{(u)}} fv\, d\Omega \approx \frac{1}{6}h^2(f_{i,j+1}v_{i,j+1} + f_{i+1,j}v_{i+1,j} + f_{i+1,j+1}v_{i+1,j+1}).$$

This gives the elementary terms of the approximation of the right-hand side on an element $T \in \mathcal{T}_h$:

$$\begin{aligned} f^h_{T_{ij}^{(l)}}(v^h) &= \frac{1}{6}h^2(f_{ij}v_{ij} + f_{i+1,j}v_{i+1,j} + f_{i+1,j+1}v_{i+1,j+1}), \\ f^h_{T_{ij}^{(u)}}(v^h) &= \frac{1}{6}h^2(f_{i,j}v_{i,j} + f_{i,j+1}v_{i,j+1} + f_{i+1,j+1}v_{i+1,j+1}). \end{aligned} \tag{2.63}$$

Summing the elementary terms like (2.61), (2.62), and (2.63) over all $T \in \mathcal{T}_h$, we obtain the approximations of the bilinear and linear forms

$$a^h(w^h, v^h) = \sum_{T \in \mathcal{T}_h} a^h_T(w^h, v^h), \tag{2.64}$$

$$f^h(v^h) = \sum_{T \in \mathcal{T}_h} f^h_T(v^h).$$

Now we come to the 'fitted' Galerkin problem: *find $u^h \in H^h$ such that*

$$a^h(u^h, v^h) = f^h(v^h) \qquad \forall\, v^h \in H^h. \tag{2.65}$$

This problem is equivalent to the system of linear algebraic equations

$$\begin{aligned} (L^h u^h)_{ij} \equiv u^h_{ij}\left(h\left(\frac{b_i \exp\sigma_i}{\exp\sigma_i - 1} + \frac{b_{i-1}}{\exp\sigma_{i-1} - 1}\right) + 2\varepsilon\right) - u^h_{i+1,j}\frac{b_i h}{\exp\sigma_i - 1} \\ - u^h_{i-1,j}\frac{b_{i-1}h\exp\sigma_{i-1}}{\exp\sigma_{i-1} - 1} - \varepsilon u^h_{i,j-1} - \varepsilon u^h_{i,j+1} \\ = f_{ij}h^2, \quad i,j = 1,2,...,n-1, \end{aligned} \tag{2.66}$$

where $u^h_{ij} = 0$ for $i = 1,...,n-1$ and $j = 0,n$ or for $j = 1,...,n-1$ and $i = 0,n$. The parameters $\{u^h_{ij}\}_{i,j=1}^{n-1}$ give the solution of the problem (2.65)

$$u^h = \sum_{i,j=1}^{n-1} u_{ij}\varphi_{ij}. \tag{2.67}$$

Enumerate the remaining unknowns and the equations in (2.66) from 1 to $(n-1)^2$ in the same way (for example, in the lexicographic order) and rewrite the system (2.66) in the vector-matrix form

$$A^h U = F \tag{2.68}$$

where

$$\begin{aligned} U &= (u^h_{1,1}, ..., u^h_{1,n-1}, u^h_{2,1}, ..., u^h_{n-1,n-1})^T, \\ F &= (f^h(\varphi_{1,1}), ..., f^h(\varphi_{1,n-1}), ..., f^h(\varphi_{n-1,n-1}))^T. \end{aligned} \tag{2.69}$$

Notice that the matrix A^h is irreducible [21], diagonal-dominant along columns and strongly diagonal-dominant along columns for $i = 0, n$. Consequently, A^h is an M-matrix and the system (2.66) satisfies the difference comparison principle and has a unique solution [21].

2.2.3 Properties of the discrete problem. The convergence result

Now we investigate the approximating properties of the discrete problem (2.65).

Lemma 20. *Let u be a solution of the problem* (2.1), (2.2) *with the conditions* (2.3), (2.4), (2.20), *and u^h be a solution of the discrete problem* (2.65). *Assume also that*

$$\varepsilon \le h. \tag{2.70}$$

Then the estimate

$$\begin{aligned} &|a^h(u^h - u^I, \varphi_{ij})| \\ &\qquad \le ch^2(\varepsilon + h + \exp(-B_1(1 - x_{i+1})/2\varepsilon)) \quad \forall\, i, j = 1, ..., n-1 \end{aligned} \tag{2.71}$$

holds.

Proof. Using the expansion (2.24) we have

$$\begin{aligned} |a^h(u^h - u^I, \varphi_{ij})| &\le |f^h(\varphi_{ij}) - f(\varphi_{ij})| + |a(u, \varphi_{ij}) - a^h(u^I, \varphi_{ij})| \\ &\le |f^h(\varphi_{ij}) - f(\varphi_{ij})| + |a(v_0, \varphi_{ij}) - a^h(v_0^I, \varphi_{ij})| \\ &+ |a(\rho_0, \varphi_{ij}) - a^h(\rho_0^I, \varphi_{ij})| + \varepsilon |a(\eta, \varphi_{ij}) - a^h(\eta^I, \varphi_{ij})|. \end{aligned} \tag{2.72}$$

Here $v_0^I, \rho_0^I, \eta^I \in H^h$ are the piecewise linear interpolants of the functions v_0, ρ_0, η and $i, j = 1, ..., n-1$.

We evaluate each term in the right-hand side of (2.72). First we estimate each expression on an elementary triangle $T \in \mathcal{T}_h$ and then we get the estimate over the whole support of φ_{ij}. It is equivalent to the estimate over $\bar{\Omega}$.

Let us take $T = T_{ij}^{(l)}$. Consider the expression

$$F_T = \frac{h^2}{6} f_{ij} - \int_T f \varphi_{ij} \, d\Omega.$$

Since $f \in C^2(\bar{\Omega})$ we use the Taylor formula

$$f(x, y) = f_{ij} + h\pi_1(x, y), \qquad |\pi_1| \le c_1 \quad \text{on} \quad T.$$

This gives

$$|F_T| = |h \int_T \pi_1 \varphi_{ij} \, d\Omega| \le \frac{1}{6} c_1 h^3.$$

The same estimate is valid on any other elementary triangle $T \in \text{supp } \varphi_{ij}$. Taking the sum over the whole support of φ_{ij}, we obtain

$$|f^s(\varphi_{ij}) - f(\varphi_{ij})| \le c_1 h^3. \tag{2.73}$$

Because of different smoothness of the solution in the x- and y-directions, we expand the bilinear forms (2.18) and (2.64) as a sum in the x- and y-directions:

$$\begin{gathered} a(u, v) = a_1(u, v) + a_2(u, v), \\ a^h(u^h, v^h) = a_1^h(u^h, v^h) + a_2^h(u^h, v^h). \end{gathered}$$

Then for the elementary bilinear forms (2.54) and (2.55) we get

$$\begin{gathered} a_T(u, v) = a_{1T}(u, v) + a_{2T}(u, v), \\ a_T^h(u^h, v^h) = a_{1T}^h(u^h, v^h) + a_{2T}^h(u^h, v^h) \end{gathered}$$

where

$$\begin{aligned} a_{1T}(u, v) &= \int_T (\varepsilon \partial_1 u - bu) \, \partial_1 v \, d\Omega, \\ a_{2T}(u, v) &= \varepsilon \int_T \partial_2 u \partial_2 v \, d\Omega, \end{aligned}$$

and

$$\begin{aligned} a_{1T}^h(w^h, v^h) &= \frac{h^2}{2} \big(\varepsilon \partial_1 w^h - b_i \left(\alpha_{1i} w^h(z_{ij}) + \alpha_{2i} w^h(z_{i+1,j})\right) \big) \partial_1 v^h, \\ a_{2T}^h(w^h, v^h) &= \frac{h^2}{2} \varepsilon \partial_2 w^h \partial_2 v^h. \end{aligned}$$

According to Lemma 15 the solution is sufficiently smooth in the y-direction, then it is easy to get the estimate of the difference

$$|a_2(u, \varphi_{ij}) - a_2^h(u^I, \varphi_{ij})|. \tag{2.74}$$

Consider the inequality

$$\begin{aligned} |a_2(u, \varphi_{ij}) - a_2^h(u^I, \varphi_{ij})| \le |a_2(u, \varphi_{ij}) - a_2(u^I, \varphi_{ij})| \\ +|a_2(u^I, \varphi_{ij}) - a_2^h(u^I, \varphi_{ij})| \end{aligned} \tag{2.75}$$

and estimate both terms in its right-hand side.

On $T = T_{ij}^{(u)}$ we have

$$A_{ij}^{(u)} = -\frac{1}{h} \int_T \varepsilon \left(\partial_2 u - \partial_2 u^I \right) \, d\Omega.$$

Use the Taylor expansion at z_{ij}

$$\begin{aligned} u(x_i, y_{j+1}) &= u(x_i, y_j) + h\partial_2 u(x_i, y_j) + h^2 \pi_1(x_i, y), \\ \partial_2 u(x, y) &= \partial_2 u(x_i, y_j) + h\pi_2(x_i, y) \end{aligned}$$

where $|\pi_1| \le c_8$ and $|\pi_2| \le c_8$ on T due to the estimate (2.21). Since u^I is the piecewise linear interpolant of u, T the equality

$$\partial_2 u^I(x, y) = \frac{u(x_i, y_{j+1}) - u(x_i, y_j)}{h} = \partial_2 u(x_i, y_j) + h\pi_1(x_i, y)$$

holds. Hence we have $|\partial_2 u - \partial_2 u^I| \le c_9 h$. Thus we obtain

$$|A_{ij}^{(u)}| \le c_{10} \varepsilon h^2.$$

The same contribution into the error comes from the triangles $T_{ij}^{(u)}$, $T_{i,j-1}^{(u)}$, $T_{i-1,j-1}^{(l)}$, and $T_{i-1,j}^{(l)}$. On the triangles $T_{ij}^{(l)}$ and $T_{i-1,j-1}^{(u)}$ the derivative $\partial_2 \varphi_{ij}$ equals zero and therefore these triangles do not make a contribution into the error. As a result, we have the following estimate of the first term in the right-hand side of (2.75):

$$|a_2(u, \varphi_{ij}) - a_2(u^I, \varphi_{ij})| \le c_{10} \varepsilon h^2.$$

Since the approximation of the second derivative gives the exact expression for linear functions, the second term in the right-hand side of (2.75) equals zero on any $T \in \mathcal{T}_h$. Finally, in the y-direction we have

$$|a_2(u, \varphi_{ij}) - a_2^h(u^I, \varphi_{ij})| \le c_{10} \varepsilon h^2. \tag{2.76}$$

To obtain the estimates in the x-direction, on a triangle $T \in \mathcal{T}_h$ we introduce the intermediate bilinear form $a_{1T}^f(u,v)$ obtained from $a_{1T}(u,v)$ by freezing the function $b(x)$ at the point x_i:

$$a_{1T}^f(u,v) = \int_T (\varepsilon\partial_1 u - b_i u)\,\partial_1 v d\Omega. \tag{2.77}$$

Then for an arbitrary function v we have the estimate

$$\begin{aligned}|a_1(v,\varphi_{ij}) - a_1^h(v^I,\varphi_{ij})| &\le |a_1(v,\varphi_{ij}) - a_1^f(v,\varphi_{ij})| \\ +|a_1^f(v,\varphi_{ij}) - a_1^f(v^I,\varphi_{ij})| &+ |a_1^f(v^I,\varphi_{ij}) - a_1^h(v^I,\varphi_{ij})|.\end{aligned} \tag{2.78}$$

First with the help of (2.78) we obtain the estimate for $v = v_0$. On $T = T_{ij}^{(l)}$ we consider

$$B_{ij}^{(l)} = a_{1T}(v_0,\varphi_{ij}) - a_{1T}^f(v_0,\varphi_{ij}) = -\frac{1}{h}\int_T (b_i - b(x))v_0\,d\Omega.$$

Expand b and v_0 in the Tailor series at x_i:

$$\begin{aligned}b(x) &= b_i + (x - x_i)b'(x_i) + h^2\pi_3(x),\\ v_0(x,y) &= v_{0,ij} + h\pi_4(x,y).\end{aligned}$$

Due to the smoothness of b and v_0, the functions $\pi_3(x)$ and $\pi_4(x,y)$ are bounded on T. As a result, we have

$$B_{ij}^{(l)} = -\frac{h^2}{3}v_{0,ij}b'(x_i) + h^3\pi_5, \qquad |\pi_5| \le c_{11}.$$

In a similar way on $T_{i-1,j}^{(l)}$ we obtain the equality

$$B_{i-1,j}^{(l)} = \frac{h^2}{3}v_{0,i-1,j}b'(x_{i-1}) + h^3\pi_6, \qquad |\pi_6| \le c_{11}$$

with the same constant c_{11} independent of ε, h, i, j. Therefore due to the smoothness of b and v_0 we have

$$|B_{ij}^{(l)} + B_{i-1,j}^{(l)}| \le 2c_{11}h^3.$$

The same contribution comes from the triangles $T_{i,j-1}^{(u)}$ and $T_{i-1,j-1}^{(u)}$. On the triangles $T_{ij}^{(u)}$ and $T_{i-1,j-1}^{(l)}$ we have $\partial_1\varphi_{ij} = 0$ and consequently these triangles do not make a contribution into the error. As a result, we get

$$|a_1(v_0,\varphi_{ij}) - a_1^f(v_0,\varphi_{ij})| \le 4c_{11}h^3. \tag{2.79}$$

Next according to (2.78) we evaluate the difference

$$|a_1^f(v_0, \varphi_{ij}) - a_1^f(v_0^I, \varphi_{ij})|. \tag{2.80}$$

On $T = T_{ij}^{(l)}$ we introduce the notation

$$C_{ij}^{(l)} = -\frac{1}{h}\int_T \left(\varepsilon(\partial_1 v_0 - \partial_1 v_0^I) - b_i(v_0 - v_0^I)\right)\, d\Omega.$$

Due to the smoothness of v_0 as well as the definition of a piecewise linear interpolant, the equality

$$\partial_1 v_0^I(x,y) = \frac{v_{0,i+1,j} - v_{0,i,j}}{h} = \partial_1 v_0(x_i, y_j) + h\pi_7(x, y_j)$$

holds. Here $|\pi_7(x, y_j)| \le c_{12}$ on T. Moreover, we have

$$\partial_1 v_0(x,y) = \partial_1 v_0(x_i, y_j) + h\pi_8(x,y)$$

where π_8 is bounded on T. Hence the estimate

$$|\partial_1 v_0 - \partial_1 v_0^I| \le c_{13} h \tag{2.81}$$

is valid. For functions v_0 and v_0^I on T use Tailor expansion in the form

$$\begin{aligned} v_0(x,y) &= v_0(x_i,y_j) + (x - x_i)\partial_1 v_0(x_i, y_j) \\ &\quad + (y - y_j)\partial_2 v_0(x_i, y_j) + h^2\pi_9(x,y) \quad \text{where} \quad |\pi_9| \le c_{14} \quad \text{on} \quad T, \end{aligned} \tag{2.82}$$

$$v_0^I(x,y) = v_0(x_i,y_j) + (x - x_i)\partial_1 v_0^I(x_i,y_j) + (y - y_j)\partial_2 v_0^I(x_i, y_j).$$

Due to the inequality (2.81) on T, the similar inequality in the y-direction

$$|\partial_2 v_0 - \partial_2 v_0^I| \le c_{15} h,$$

and (2.82) we get the estimate

$$|v_0 - v_0^I| \le c_{16} h^2.$$

Then we have

$$|C_{ij}^{(l)}| \le c_{17} h^2(\varepsilon + h). \tag{2.83}$$

The same contribution comes from the triangles $T_{i-1,j}^{(l)}$, $T_{i,j-1}^{(u)}$, and $T_{i-1,j-1}^{(u)}$. The triangles $T_{ij}^{(u)}$ and $T_{i-1,j-1}^{(l)}$ do not make a the contribution into the error because of equality $\partial_1 \varphi_{ij} = 0$.

From (2.83) we obtain

$$|a_1^f(v_0,\varphi_{ij}) - a_1^f(v_0^I,\varphi_{ij})| \le 4c_{18}(\varepsilon+h)h^2. \tag{2.84}$$

To complete the proof of the estimate (2.78) for $v=v_0$, we evaluate the term $|a_1^f(v_0^I,\varphi_{ij}) - a_1^h(v_0^I,\varphi_{ij})|$. On $T=T_{ij}^{(l)}$ we have

$$\begin{aligned} D_{ij}^{(l)} &= \frac{b_i}{h}(\int_T v_0^I\,d\Omega - \frac{h^2}{2}(\alpha_{1i}v_{0,i,j}+\alpha_{2i}v_{0,i+1,j})) \\ &= \frac{b_i h}{6}(v_{0,i,j}+v_{0,i+1,j}+v_{0,i+1,j+1}-3\alpha_{1i}v_{0,i,j}-3\alpha_{2i}v_{0,i+1,j}). \end{aligned}$$

Use the Taylor expansion (2.82) for $v_{0,i+1,j}$ and $v_{0,i+1,j+1}$ near (x_i,y_j). Since $\alpha_{1i}+\alpha_{2i}=1$, we get

$$D_{ij}^{(l)} = \frac{b_i h^2}{6}((2-3\alpha_{2i})\partial_1 v_{0,i,j}+\partial_2 v_{0,i,j})+h^3\pi_{9,ij}$$

where $\pi_{9,ij}$ is a function bounded on T. In a similar way on $T_{i-1,j}^{(l)}$ we obtain the expression

$$D_{i-1,j}^{(l)} = -\frac{b_i h^2}{6}\,((2-3\alpha_{2,i-1})\partial_1 v_{0,i-1,j}+\partial_2 v_{0,i-1,j})+h^3\pi_{9,i-1,j}$$

with the value $\pi_{9,i-1,j}$ of the function π_9 bounded on T.

Consider the function

$$\widetilde{\alpha}(t) = t\left(\frac{\exp(t)}{(\exp(t)-1)^2}-\frac{1}{t^2}\right).$$

This function approaches zero as $t\to 0$ or $t\to+\infty$. Since it is continuous on the interval $(0,\infty)$, it is bounded

$$|\widetilde{\alpha}(t)| \le c_{19} \qquad \text{on} \quad (0,\infty)$$

with a constant c_{19} independent of h, ε, x, t. Taking into account

$$\partial_1\alpha_2(x) = \frac{b_1'(x)}{b_1(x)}\widetilde{\alpha}(hb_1(x)/\varepsilon),$$

(2.4), and the smoothness of b and its derivative, we get

$$|\partial_1\alpha_2(x)| \le c_{19}\|b'\|_\infty/B_1.$$

Since the functions b, $\alpha_2(x)$, $\partial_1 v_0$, $\partial_2 v_0$ have bounded derivatives with respect to x, by the mean value theorem we get

$$|D_{ij}^{(l)} + D_{i-1,j}^{(l)}| \le c_{20} h^3.$$

The same contribution comes from the triangles $T_{i,j-1}^{(u)}$, $T_{i-1,j-1}^{(u)}$. The triangles $T_{ij}^{(u)}$ and $T_{i-1,j-1}^{(l)}$ do not make a contribution into the error because of the equality $\partial_1 \varphi_{ij} = 0$. As a result, we get

$$|a_1^f(v_0^I, \varphi_{ij}) - a_1^h(v_0^I, \varphi_{ij})| \le 2c_{20} h^3. \tag{2.85}$$

From (2.78) together with (2.79), (2.84), and (2.85) we obtain the estimate of the second term in (2.72)

$$|a_1(v_0, \varphi_{ij}) - a_1^h(v_0^I, \varphi_{ij})| \le 2c_{21} h^2 (h + \varepsilon). \tag{2.86}$$

Now let us obtain the similar estimates for the pair ρ_0 and ρ_0^I. To do this, we consider the third term in (2.72). On $T = T_{ij}^{(l)}$ we have

$$E_{ij}^{(l)} = a_T(\rho_0, \varphi_{ij}) - a_T^f(\rho_0, \varphi_{ij}) = -\frac{1}{h} \int_T (b_i - b(x)) \rho_0 \, d\Omega.$$

Since $|b_i - b(x)| \le h \|b'\|_\infty$ on $[x_i, x_{i+1}]$, we get

$$|E_{ij}^{(l)}| \le c_{22} \int_T |\rho_0| \, d\Omega. \tag{2.87}$$

Let us examine two variants of the behavior of the function $g(y) = -v_0(1, y)$. First assume that $g(y)$ changes its sign on the segment $[y_j, y_{j+1}]$. It involves $|g(y)| \le h \|g'\|_\infty$ on $[y_j, y_{j+1}]$. Therefore $|\rho_0| \le c_{23} h$ on T and we have

$$|E_{ij}^{(l)}| \le c_{24} h^3. \tag{2.88}$$

Next we assume that $g(y)$ does not change its sign on the segment $[y_j, y_{j+1}]$ and is, for example, nonnegative. Then by the mean value theorem we get

$$\int_T |\rho_0| \, d\Omega = \int_T \rho_0 \, d\Omega = \frac{h^2}{2} \rho_0(\tau^*, y^*) \tag{2.89}$$

where $(\tau^*, y^*) \in T$. Because of (2.4) we obtain

$$\rho_0(\tau^*, y^*) \le c_{25} \exp(-(1 - x_{i+1}) B_1 / \varepsilon). \tag{2.90}$$

Combining (2.87), (2.88)–(2.90), we get the estimate

$$|E_{ij}^{(l)}| \le c_{26} h^2 (h + \exp(-(1 - x_{i+1}) B_1/\varepsilon)).$$

The same contribution comes from the triangles $T_{i-1,j}^{(l)}$, $T_{i,j-1}^{(u)}$, and $T_{i-1,j-1}^{(u)}$. On the triangles $T_{ij}^{(u)}$, $T_{i-1,j-1}^{(l)}$ we have $\partial_1 \varphi_{ij} = 0$ and consequently the contribution of these triangles equals zero. As a result, we have

$$|a(\rho_0, \varphi_{ij}) - a^f(\rho_0, \varphi_{ij})| \le c_{27} h^2 (h + \exp(-(1 - x_{i+1}) B_1/\varepsilon)). \tag{2.91}$$

Now we evaluate the difference $a_1^f(\rho_0, \varphi_{ij}) - a_1^h(\rho_0^I, \varphi_{ij})$. Let us take $T = T_{ij}^{(l)}$ and introduce the function

$$\hat{\rho}(x, y) = g^I(y) \exp(-(1 - x) b_i/\varepsilon)$$

where g^I is the piecewise linear interpolant of g. According to the construction of the quadrature rule, we have

$$a_1^f(\hat{\rho}, \varphi_{ij}) = a_1^h(\rho_0^I, \varphi_{ij}).$$

Thus, on $T_{ij}^{(l)}$ we obtain the representation

$$\begin{aligned} G_{ij}^{(l)} &= a_{1T}^f(\rho_0, \varphi_{ij}) - a_{1T}^h(\rho_0^I, \varphi_{ij}) = a_{1T}^f(\rho_0, \varphi_{ij}) - a_{1T}^f(\hat{\rho}, \varphi_{ij}) \\ &= -\frac{1}{h} \int_T (\varepsilon \partial_1 (\rho_0 - \hat{\rho}) - b_i (\rho_0 - \hat{\rho}))\, d\Omega. \end{aligned} \tag{2.92}$$

Taking into account the behavior of the functions ρ_0 and $\hat{\rho}$ we conclude that the estimate of G_{ij} is worst near $x = 1$. Consider this case in detail. Assume that $x_i \ge 2/3$. Then we get

$$\begin{aligned} |\rho_0 - \hat{\rho}| &= |g(y) \exp(-(1 - x) b(x)/\varepsilon) - g^I(y) \exp(-(1 - x) b_i/\varepsilon)| \\ &\le |g(y) \left(\exp(-(1 - x) b(x)/\varepsilon) - \exp(-(1 - x) b_i/\varepsilon)\right)| \\ &\quad + |g(y) - g^I(y)| \exp\left(-(1 - x) b_i/\varepsilon\right). \end{aligned} \tag{2.93}$$

To estimate the first term we take into account the fact that the function $t \exp(-t)$ is bounded for $t \in [0, \infty)$ and use the mean value theorem. Then we obtain the upper bound

$$c_{28} h \exp\left(-(1 - x_{i+1}) B_1/2\varepsilon\right).$$

Since the interpolant g^I approximates g with the second order accuracy, we have the estimate of the second term in (2.93):

$$|\rho_0 - \hat{\rho}| \le c_{29} h \left(h + \exp\left(-(1 - x_{i+1}) B_1/2\varepsilon\right)\right). \tag{2.94}$$

Using similar reasoning for the first derivative, we get

$$\begin{aligned}
|\partial_1\rho_0 - \partial_1\hat{\rho}| &= \left| g(y)\left(\frac{b(x)}{\varepsilon} - b'(x)\frac{1-x}{\varepsilon}\right)\exp\left(-(1-x)b(x)/\varepsilon\right)\right. \\
&\quad \left. - g^I(y)\frac{b_i}{\varepsilon}\exp\left(-(1-x)b_i/\varepsilon\right)\right| \\
&\le \frac{b(x)}{\varepsilon}g(y)\left|\exp\left(-(1-x)b(x)/\varepsilon\right) - \exp\left(-(1-x)b_i/\varepsilon\right)\right| \\
&\quad + \left|\frac{1}{\varepsilon}\left(g(y)b(x) - g^I(y)b_i\right)\right|\exp\left(-(1-x)b_i/\varepsilon\right) \\
&\quad + \left|g(y)b'(x)\frac{1-x}{\varepsilon}\right|\exp\left(-(1-x)b(x)/\varepsilon\right) \\
&\le c_{30}\frac{h}{\varepsilon}\exp\left(-(1-x_{i+1})B_1/2\varepsilon\right).
\end{aligned} \tag{2.95}$$

Combining (2.94) and (2.95), we have

$$|G_{ij}^{(l)}| \le c_{31}h^2\left(h + \exp(-(1-x_{i+1})B_1/2\varepsilon)\right) \quad \text{for} \quad x_i \ge 2/3. \tag{2.96}$$

Now assume that $x_i < 2/3$. Then we have

$$\exp(-(1-x)b_1(x^*)/\varepsilon) \le c_{32}\varepsilon^2 \le c_{33}h^2 \quad \text{for all} \quad x^* \in [x_i, x_{i+1}].$$

This gives

$$\begin{aligned}
|\rho_0 - \hat{\rho}| &\le |g(y)|s(x)\exp(-(1-x)b(x)/\varepsilon) \\
&\quad + |g^I(y)|s(x)\exp(-(1-x)b_i/\varepsilon) \le c_{34}h^2, \\
|\partial_1\rho_0 - \partial_1\hat{\rho}| &\le c_{35}\varepsilon^{-1}\big(|g(y)|s(x)\exp(-(1-x)b(x)/\varepsilon) \\
&\quad + |g^I(y)|s(x)\exp(-(1-x)b_i/\varepsilon)\big) \le c_{36}h.
\end{aligned}$$

Therefore for $x_i < 2/3$ the following estimate is valid:

$$|G_{ij}^{(l)}| \le c_{37}h^3.$$

Thus, (2.96) holds for all $x_i \in (0,1)$. The same estimates are valid for the triangles $T_{i-1,j}^{(l)}$, $T_{i,j-1}^{(u)}$, $T_{i-1,j-1}^{(u)}$. The contribution from the triangles $T_{ij}^{(u)}$ and $T_{i-1,j-1}^{(l)}$ is zero since the derivative $\partial_1\varphi_{ij}$ equals zero on these triangles.

Therefore we obtain

$$|a_1^f(\rho_0, \varphi_{ij}) - a_1^h(\rho_0^I, \varphi_{ij})| \le c_{38}h^2(h + \exp(-(1-x_{i+1})B_1/2\varepsilon). \tag{2.97}$$

Finally, we need to obtain the estimate for the last term in the right-hand side of (2.72) for the functions η and η^I:

$$\left|a_1(\eta, \varphi_{ij}) - a_1^h(\eta^I, \varphi_{ij})\right|.$$

On $T = T_{ij}^{(l)}$ we consider

$$H_{ij}^{(l)} = a_{1T}(\eta, \varphi_{ij}) - a_{1T}^f(\eta, \varphi_{ij}) = -\frac{1}{h}\int_T (b_i - b(x))\eta\, d\Omega.$$

Because η is bounded on T according to Lemma 19 and b is sufficiently smooth, we get

$$\left|H_{ij}^{(l)}\right| \le c_{39}h^2.$$

The contribution from the other triangles with the vertex $z_{ij} = (x_i, y_j)$ has the same order. Therefore we have

$$\left|a_1(\eta, \varphi_{ij}) - a_1^f(\eta, \varphi_{ij})\right| \le 4c_{39}h^2.$$

Now we estimate the difference $\left|a_1^f(\eta, \varphi_{ij}) - a_1^f(\eta^I, \varphi_{ij})\right|$. On $T = T_{ij}^{(l)}$ we have

$$I_{ij}^{(l)} = -\frac{1}{h}\int_T \varepsilon(\partial_1\eta - \partial_1\eta^I) - b_i(\eta - \eta^I))\, d\Omega.$$

Due to (2.44) we get

$$|\partial_1\eta| \le c_{40}\left(1 + \varepsilon^{-1}\exp\left(-B_1(1 - x_{i+1})/2\varepsilon\right)\right) \qquad \text{on} \quad T. \tag{2.98}$$

Because of the Lagrange theorem $|\partial_1\eta^I(x,y)| = |\partial_1\eta(t, y_j)|$ on T. It involves the same estimate as (2.98) for both expressions. Under (2.70) we obtain

$$\varepsilon|\partial_1\eta - \partial_1\eta^I| \le c_{41}\varepsilon\left(1 + \varepsilon^{-1}\exp\left(-B_1(1 - x_{i+1})/2\varepsilon\right)\right). \tag{2.99}$$

Further, in order to estimate the difference $\eta - \eta^I$ on T we use the Taylor formula for η in the form

$$\eta(x,y) = \eta_{ij} + (y - y_j)\partial_2\eta(x_i, y_j) + h\pi_{10}(x, y_j). \tag{2.100}$$

Here because of (2.44) and (2.45) the function π_{10} satisfies the inequality

$$|\pi_{10}| \le c_{42}\left(1 + h\varepsilon^{-1} + \varepsilon^{-1}\exp\left(-B_1(1 - x_{i+1})/2\varepsilon\right)\right) \tag{2.101}$$

on T. Moreover, we have

$$\eta^I(x,y) = \eta_{ij} + (x - x_i)\frac{\eta_{i+1,j} - \eta_{ij}}{h} + (y - y_j)\frac{\eta_{i,j+1} - \eta_{ij}}{h} \quad \text{on} \quad T.$$

Using (2.100) for $\eta_{i+1,j}$ and $\eta_{i,j+1}$, we obtain

$$|\eta - \eta^I| \le c_{43} h \left(1 + \varepsilon^{-1} h + \varepsilon^{-1} \exp\left(-B_1(1 - x_{i+1})/2\varepsilon\right)\right). \tag{2.102}$$

Combining (2.99) and (2.102) we get on T

$$|I_{ij}^{(l)}| \le c_{44} \frac{h^2}{\varepsilon} \left(\varepsilon + h + \exp\left(-B_1(1 - x_{i+1})/2\varepsilon\right)\right). \tag{2.103}$$

The same estimate is valid for the triangles $T_{i-1,j}^{(l)}$, $T_{i,j-1}^{(u)}$, $T_{i-1,j-1}^{(u)}$. The contribution from the triangles $T_{ij}^{(u)}$ and $T_{i-1,j-1}^{(l)}$ equals zero. Therefore we have

$$|a_1^f(\eta, \varphi_{ij}) - a_1^f(\eta^I, \varphi_{ij})| \le c_{45} \frac{h^2}{\varepsilon} \left(\varepsilon + h + \exp\left(-B_1(1 - x_{i+1})/2\varepsilon\right)\right). \tag{2.104}$$

It remains to evaluate the difference

$$|a_1^f(\eta^I, \varphi_{ij}) - a_1^h(\eta^I, \varphi_{ij})|. \tag{2.105}$$

On $T = T_{ij}^{(l)}$ we have

$$\begin{aligned} J_{ij}^{(l)} &= -\frac{b_i}{h}\Big(\int_T \eta^I \, d\Omega - \frac{h^2}{2}(\alpha_{1i}\eta_{ij} + \alpha_{2i}\eta_{i+1,j})\Big) \\ &= -\frac{b_{1i} h}{6}(\eta_{ij} + \eta_{i+1,j} + \eta_{i+1,j+1} - 3\alpha_{1i}\eta_{ij} - 3\alpha_{2i}\eta_{i+1,j}). \end{aligned}$$

Use the Taylor formula in the form (2.100) for $\eta_{i+1,j}$ and $\eta_{i+1,j+1}$ and the equality $\alpha_{1i} + \alpha_{2i} = 1$. Since α_{1i}, α_{2i}, and $b(x)$ are bounded, we have

$$J_{ij}^{(l)} = \frac{b_i h^2}{6} \partial_2 \eta_{ij} + h^2 \pi_{11}(x, y)$$

with the estimate of $\pi_{11}(x, y)$ similar to (2.101). Considering (2.105) on the triangle $T_{i,j-1}^{(l)}$, we get

$$|J_{ij}^{(u)} + J_{ij-1}^{(u)}| \le c_{46} \frac{h^2}{\varepsilon} \left(\varepsilon + h + \exp\left(-B_1(1 - x_{i+1})/2\varepsilon\right)\right).$$

The triangles $T_{i-1,j-1}^{(u)}$ and $T_{i,j-1}^{(u)}$ make the same contribution. The contribution of the triangles $T_{ij}^{(u)}$ and $T_{i-1,j-1}^{(l)}$ equals zero.

Therefore for (2.105) we get

$$\begin{aligned} &\left| a_{1T}^f(\eta^I, \varphi_{ij}) - a_{1T}^h(\eta^I, \varphi_{ij}) \right| \\ &\qquad \le c_{47} \frac{h^2}{\varepsilon} \left(\varepsilon + h + \exp\left(-B_1(1 - x_{i+1})/2\varepsilon\right)\right). \end{aligned} \tag{2.106}$$

Combining (2.73), (2.76), (2.86), (2.97), and (2.106), due to (2.70) we obtain the estimate (2.71).□

The next result gives the barrier functions to estimate the right-hand side of (2.71).

Lemma 21. *Let us assume that*

$$\varepsilon \le c_1 h \tag{2.107}$$

with a constant c_1*. There exist the mesh functions* φ^h *and* ψ^h *on* $\bar{\Omega}_h$ *with the properties*

$$|\varphi^h| \le c_2 \quad in \quad \Omega_h, \tag{2.108}$$
$$|\psi^h| \le c_3 h \quad in \quad \Omega_h \tag{2.109}$$

such that

$$L^h \varphi^h \ge h^2 \quad in \quad \Omega_h, \tag{2.110}$$
$$\varphi^h \ge 0 \quad on \quad \Gamma_h \tag{2.111}$$

and

$$L^h \psi^h \ge h^2 \exp\left(-B_1(1-x_{i+1})/2\varepsilon\right) \quad in \quad \Omega_h, \tag{2.112}$$
$$\psi^h \ge 0 \quad on \quad \Gamma_h. \tag{2.113}$$

Proof. Consider the expression

$$\frac{1}{h}\left(\frac{b_i \exp(s_i)}{\exp(s_i)-1} - \frac{b_{i-1}\exp(s_{i-1})}{\exp(s_{i-1})-1}\right) \tag{2.114}$$

where $s_i = b_i h/\varepsilon$. It can be thought as the difference of the values of the function $f(s) = (s\exp(s))/(\exp(s)-1)$ at neighboring nodes of the grid. Then by the mean value theorem we have

$$\begin{aligned} \frac{\varepsilon}{h^2}\left|\frac{s_i \exp(s_i)}{\exp(s_i)-1} - \frac{s_{i-1}\exp(s_{i-1})}{\exp(s_{i-1})-1}\right| &\le \frac{\varepsilon}{h^2} f'(s^*)|s_i - s_{i-1}| \\ &\le f'(s^*)b'(x^*) \le c_5 \quad \text{where} \quad s^* \in \left[\frac{B_1 h}{\varepsilon}, \frac{B_2 h}{\varepsilon}\right]. \end{aligned} \tag{2.115}$$

We take into account the estimate

$$|s_i - s_{i-1}| = \frac{h}{\varepsilon}|b_i - b_{i-1}| \le \frac{h}{\varepsilon} b'(x^*)h, \quad x^* \in [x_{i-1}, x_i].$$

Thus, the difference (2.114) is bounded.

Now we consider the expression

$$\frac{1}{h}\left(\frac{b_i}{\exp(s_i)-1}-\frac{b_{i-1}}{\exp(s_{i-1})-1}\right). \tag{2.116}$$

In a similar way we introduce the function $f(s) = s/(\exp(s)-1)$. Using the mean value theorem we get the estimate for (2.116)

$$\begin{aligned}\frac{\varepsilon}{h^2}\left|\frac{s_i}{\exp(s_i)-1}-\frac{s_{i-1}}{\exp(s_{i-1})-1}\right| &\le \frac{\varepsilon}{h^2} f'(s^{**})|s_i - s_{i-1}| \\ \le f'(s^{**})b'(x^{**}) \le c_6 \quad \text{where} \quad s^{**} &\in \left[\frac{B_1 h}{\varepsilon}, \frac{B_2 h}{\varepsilon}\right], \quad x^{**} \in [x_{i-1}, x_i].\end{aligned} \tag{2.117}$$

Thus, the difference in (2.116) is also bounded.

Put $\sigma = 4(c_5 + c_6)/B_1$ and introduce the function φ^h by

$$\varphi^h_{0,j} = 0, \quad \frac{\varphi^h_{i,j} - \varphi^h_{i-1,j}}{h} = \sigma\exp(\sigma x_i), \quad i = 1, ..., n; \;\; \forall\, j = 1, ..., n-1.$$

We want to show that φ^h satisfies the conditions (2.110), (2.111). Rewrite $\dfrac{1}{h^2} L^h\varphi^h$ in the form

$$\begin{aligned}-\frac{b_{i-1}\exp(s_{i-1})}{h\,(\exp(s_{i-1})-1)}\varphi^h_{i-1,j} + \left(\frac{b_{i-1}}{h\,(\exp(s_{i-1})-1)} + \frac{b_i\exp(s_i)}{h\,(\exp(s_i)-1)}\right)\varphi^h_{ij} \\ - \frac{b_i}{h\,(\exp(s_i)-1)}\varphi^h_{i+1,j}.\end{aligned} \tag{2.118}$$

Rearranging the terms in (2.118), we have

$$\begin{aligned}&\frac{b_i\exp(s_i)}{h\,(\exp(s_i)-1)}\left(\varphi^h_{ij} - \varphi^h_{i-1,j}\right) - \frac{b_i}{h\,(\exp(s_i)-1)}\left(\varphi^h_{i+1,j} - \varphi^h_{ij}\right) \\ &+\frac{1}{h}\left(\frac{b_i\exp(s_i)}{\exp(s_i)-1} - \frac{b_{i-1}\exp(s_{i-1})}{\exp(s_{i-1})-1}\right)\varphi^h_{i-1,j} \\ &-\frac{1}{h}\left(\frac{b_i}{\exp(s_i)-1} - \frac{b_{i-1}}{\exp(s_{i-1})-1}\right)\varphi^h_{i0}.\end{aligned} \tag{2.119}$$

Take into consideration the inequalities $\exp(-b_i h/\varepsilon) \le \exp(-B_1 h/\varepsilon) \le 1/2$ for $c_1 = B_1/\ln(1/2)$ in (2.107) and $\exp(\sigma h) \ge 1$. Then the difference of two first terms in (2.119) has the lower estimate

$$\frac{b_i \exp(s_i)}{\exp(s_i)-1}\sigma \exp(\sigma x_i) - \frac{b_i}{\exp(s_i)-1}\sigma \exp(\sigma x_{i+1})$$
$$= \frac{b_i \exp(s_i)}{\exp(s_i)-1}\sigma \left(\exp(\sigma x_i) - \exp\left(\sigma x_i + h\sigma - b_i h/\varepsilon\right)\right)$$
$$\geq \frac{1}{2}\sigma b_i \exp(\sigma x_i) \geq \frac{1}{2}\sigma B_1 \exp(\sigma x_i).$$

In view of the definition of σ the first term in (2.119) is twice as much as the remaining two terms, hence the conditions (2.110), (2.111) are valid.

Show that the condition (2.108) holds for the function φ^h. From the definition of φ^h we have

$$\frac{\varphi^h_{i,j} - \varphi^h_{0,j}}{h} = \sigma \sum_{k=0}^{i} \exp(\sigma x_k) = \sigma \exp(\sigma x_i) \sum_{k=0}^{i} \exp(-k\sigma h).$$

Then φ^h has the following representation

$$\varphi^h_{ij} = h\sigma \exp(\sigma x_i) \sum_{k=0}^{i} \exp(-k\sigma h).$$

The sum in this expression is the partial sum of a geometric progression, hence we get the estimate

$$\varphi^h_{ij} \leq \frac{h\sigma \exp(\sigma x_i)}{1-\exp(-\sigma h)}.$$

For sufficiently small h the following inequality holds:

$$\frac{\sigma h}{1-\exp(-\sigma h)} \leq 2.$$

It can be proved using the Taylor expansion of the function $f(t) = 1 - \exp(-t)$ at zero. Then we have $\varphi^h_{ij} \leq 2\exp(\sigma x_i)$ on Ω_h. The proof of the properties of φ^h_{ij} is complete.

Now consider the function ψ^h defined by

$$\frac{\psi^h_{ij} - \psi^h_{i-1,j}}{h} = \exp(-B_1(1-x_{i+1})/2\varepsilon) \quad \forall\, j = 0,1,...,n; \quad \psi^h_{i,0} = 0.$$

In order to prove the properties (2.112), (2.113) we consider

$$\frac{1}{h^2}L^h\psi^h = -\frac{b_{i-1}\exp(s_{i-1})}{h\left(\exp(s_{i-1})-1\right)}\psi^h_{i-1,j}$$
$$+\left(\frac{b_{i-1}}{h\left(\exp(s_{i-1})-1\right)} + \frac{b_i \exp(s_i)}{h\left(\exp(s_i)-1\right)}\right)\psi^h_{ij} - \frac{b_i}{h\left(\exp(s_i)-1\right)}\psi^h_{i+1,j}. \tag{2.120}$$

For convenience we rearrange terms:

$$
\begin{aligned}
&\frac{b_i \exp(s_i)}{h\left(\exp(s_i)-1\right)}\left(\psi^h_{ij}-\psi^h_{i-1,j}\right)-\frac{b_i}{h\left(\exp(s_i)-1\right)}\left(\psi^h_{i+1,j}-\psi^h_{ij}\right)\\
&+\frac{1}{h}\left(\frac{b_i \exp(s_i)}{\exp(s_i)-1}-\frac{b_{i-1}\exp(s_{i-1})}{\exp(s_{i-1})-1}\right)\psi^h_{i-1,j}\\
&-\frac{1}{h}\left(\frac{b_i}{\exp(s_i)-1}-\frac{b_i}{\exp(s_i)-1}\right)\psi^h_{i}.
\end{aligned}
\tag{2.121}
$$

Take into consideration the inequality $\exp\left(-B_1 h/2\varepsilon\right) \le 1/2$ which holds for $\varepsilon \le h\, B_1/\ln 4$. Then we estimate the difference of the first two terms in (2.121):

$$
\begin{aligned}
&\frac{b_i \exp(s_i)}{\exp(s_i)-1}\exp(-B_1(1-x_{i+1})/2\varepsilon)-\frac{b_i}{\exp(s_i)-1}\exp(-B_1(1-x_{i+2})/2\varepsilon)\\
&=\frac{b_i \exp(s_i)}{\exp(s_i)-1}\exp(-B_1(1-x_{i+1})/2\varepsilon)\left(1-\exp\left(\frac{B_1 h}{2\varepsilon}-\frac{b_i h}{\varepsilon}\right)\right)\\
&\ge \frac{1}{2} b_i \exp(-B_1(1-x_{i+1})/2\varepsilon) \ge c_{10}\exp(-B_1(1-x_{i+1})/2\varepsilon)
\end{aligned}
$$

with the constant $c_{10} = B_1/2$.

The coefficients of $\psi^h_{i-1,j}$ and $\psi^h_{i+1,j}$ are bounded on Ω_h due to the estimates (2.115) and (2.117), respectively. Since the functions $\psi^h_{i-1,j}$ and $\psi^h_{i+1,j}$ themselves are of the first order with respect to h, we obtain the estimate (2.112).

Next we examine the condition (2.109) for ψ^h. In the same way as for the function φ^h, we have

$$
\begin{aligned}
\psi^h_{ij} &= \psi^h_{0j} + h\exp(-B_1(1-x_{i+1})/2\varepsilon)\sum_{k=0}^{i}\exp(-kB_1 h)/2\varepsilon)\\
&\le h\exp(-B_1(1-x_{i+1})/2\varepsilon)\frac{1}{1-\exp(-B_1 h/2\varepsilon)}\\
&\le c_8 h\exp(-B_1(1-x_{i+1})/2\varepsilon) \le c_9 h \quad \text{on} \quad \Omega_h.
\end{aligned}
$$

Thus, the function ψ^h satisfies (2.109). This completes the proof. □

Finally, using Lemmata 20 and 21, we formulate the main result.

Theorem 22. *Assume that* (2.4), (2.20) *hold. Then there exist constants* h_0 *and* c_1 *independent of* h *and* ε *such that* $\forall\ h \le h_0$ *and for* $\varepsilon \le h$ *the solution* u^h *of the problem* (2.65) *satisfies the estimate*

$$
\max_{\bar{\Omega}_h}|u-u^h| = \|u^I-u^h\|_{\infty,h} \le c_1 h \tag{2.122}
$$

where u is the solution of the problem (2.18), (2.19).

Proof. Introduce the function

$$\phi^h = c_2 h \varphi^h + c_3 \psi^h$$

with φ^h and ψ^h from Lemma 21. From (2.110) – (2.113) we get

$$\begin{aligned} |L^h \phi^h| \ &\geq \ h^2 \left(h + \exp\left(-B_1(1 - x_{i+1})/2\varepsilon\right)\right) \quad \text{in} \quad \Omega_h, \\ |\phi^h| &\geq 0 \quad \text{on} \quad \Gamma_h. \end{aligned}$$

Then by Lemma 21 the inequality

$$|\phi^h| \leq c_4 h \quad \text{on} \quad \bar{\Omega}_h$$

holds. In view of the definition (2.66) of the operator L^h and by Lemma 20 we have

$$\begin{aligned} |L^h(u^h - u^I)_{ij}| &\leq c_5 h^2 (\varepsilon + h + \exp(-(1 - x_{i+1}) B_1/2\varepsilon)) \\ &\leq L^h(\phi^h)_{ij} \quad \forall\, i, j = 1, ..., n-1. \end{aligned}$$

Therefore

$$|u^h(x_i, y_j) - u(x_i, y_j)| \leq c_6 h$$

for $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$. For $i = 0, n$ and $j = 0, ..., n$ or for $j = 0, n$ and $i = 0, ..., n$ the difference $u^h(x_i, y_j) - u(x_i, y_j)$ equals zero. Since $u^I(x_i, y_j) = u(x_i, y_j)$, we have (2.122). $\square$

2.3 Construction of the method for the problem with regular and parabolic boundary layers

In this section we reject the restriction (2.20) and consider the convection-diffusion problem whose solution has regular and parabolic boundary layers.

2.3.1 Properties of the differential problem.

Consider the problem (2.1), (2.2) under the conditions (2.3) – (2.5).

To describe the behaviour of the solution for small ε, we use the following expansion of the solution

$$u = u_0 + \rho_0 + \varepsilon \eta \quad \text{on} \quad \bar{\Omega}. \tag{2.123}$$

Here u_0 is the solution of the 'partially reduced' problem

$$L_{par} u_0 \equiv -\varepsilon \partial_{22} u_0 + \partial_1 (b u_0) = f \quad \text{in} \quad \Omega, \tag{2.124}$$

$$u_0 = 0 \quad \text{on} \quad \Gamma_{in} \cup \Gamma_{tg}. \tag{2.125}$$

The function ρ_0 is the regular boundary layer component

$$\rho_0(x,y) = g(y)s(x)\exp\left(-(1-x)b(x)/\varepsilon\right) \tag{2.126}$$

where

$$g(y) = -u_0(1,y) \qquad \text{on} \quad \Gamma_{out}. \tag{2.127}$$

The cut-off function $s(x) \in C^4[0,1]$ was introduced in (1.9). The function u_0 in (2.123), unlike the analogous component in (2.24), is not smooth in the y-direction. But it still is sufficiently smooth in the x-direction.

The operator L_{par} satisfies the comparison principle. We formulate it for the family of differential operators

$$\mathcal{L}_{par}v \equiv -\varepsilon\partial_{22}v + b\partial_1 v + dv \tag{2.128}$$

where $b(x)$ satisfies the conditions (2.3), (2.4) and $d(x)$ is a sufficiently smooth bounded function which will be specified further in each individual case.

Lemma 23. *Assume that $\varepsilon > 0$ and* (2.3), (2.4) *hold. Assume also that $u, w \in C^2(\bar{\Omega})$ satisfy the inequalities*

$$|\mathcal{L}_{par}u| \le \mathcal{L}_{par}w \quad in \quad \Omega, \qquad |u| \le w \quad on \quad \Gamma_{in} \cup \Gamma_{tg}. \tag{2.129}$$

Then we have

$$|u| \le w \qquad on \quad \bar{\Omega}. \tag{2.130}$$

The lemma can be proved in the same way as Lemma 13 for the operator $\mathcal{L}$.

The following lemmata give some estimates of the functions from (2.123) and of its derivatives, that are requred to construct and to investigate the discrete problem.

Lemma 24. *Assume that $\varepsilon > 0$ is sufficiently small and* (2.3), (2.4) *hold. Then the estimate*

$$\left|\frac{\partial^j u(x,y)}{\partial y^j}\right| \le c_1\left(1 + \varepsilon^{-j/2}B(y)\right), \quad j = 1,2,3, \quad on \quad \bar{\Omega} \tag{2.131}$$

is valid where $B(y) = \exp\left(-\gamma y/\sqrt{\varepsilon}\right) + \exp\left(-\gamma(1-y)/\sqrt{\varepsilon}\right)$ with a constant $\gamma > 0$.

Proof. Set $d(x) = b'(x)$ for the operator $\mathcal{L}$ introduced by (2.6). Let σ be defined by (2.10). Assume that ε satisfies the condition (2.12). For the barrier function

$$w(x,y) = c_2\left(1 - \exp\left(-\gamma y/\sqrt{\varepsilon}\right)\right)\exp(\sigma x)$$

we put $\gamma = \sqrt{B_1/2}$ and $c_2 = 2\|f\|_\infty/B_1$. According to (2.13) we have

$$\begin{aligned}(\mathcal{L}w)(x,y) &= c_2\gamma^2 \exp\left(-\gamma y/\sqrt{\varepsilon}\right)\exp(\sigma x) + \left(-\varepsilon\sigma^2 + \sigma b(x) + d(x)\right) w(x,y) \\ &\geq |(\mathcal{L}u)(x,y)| \quad \text{for} \quad (x,y) \in \Omega, \\ w(x,y) &\geq |u(x,y)| = 0 \quad \text{for} \quad (x,y) \in \Gamma.\end{aligned}$$

Thus, applying Lemma 13 we see that the solution u is bounded on $\bar{\Omega}$.

Besides, due to the equality $w(x,0) = 0$ on the boundary $y = 0$ the estimate

$$|\partial_2 u(x,0)| \leq \partial_2 w(x,0) \leq c_3\varepsilon^{-1/2}. \tag{2.132}$$

holds. To prove the same estimate on the boundary $y = 1$ we take

$$w(x,y) = c_4\left(1 - \exp\left(-\gamma(1-y)/\sqrt{\varepsilon}\right)\right)\exp(\sigma x)$$

as the barrier function and use Lemma 13. In a similar way as (2.132), we get

$$|\partial_2 u(x,y)| \leq \partial_2 w(x,y) \leq c_5\varepsilon^{-1/2} \quad \text{for} \quad (x,y) \in \Gamma_{tg}. \tag{2.133}$$

In addition, because of (2.2) we have

$$|\partial_2 u(x,y)| = 0 \quad \text{for} \quad (x,y) \in \Gamma_{in} \cup \Gamma_{out}. \tag{2.134}$$

To prove the estimate (2.131) for $j = 1$ we differentiate the equation (2.1) with respect to y. We introduce the notation $v_1 = \partial_2 u$. Then we get

$$\mathcal{L}v_1 = \partial_2 f \quad \text{in} \quad \Omega.$$

Choosing a sufficiently large positive constant c_6 we see that the barrier function

$$w(x,y) = c_6\left(1 + \varepsilon^{-1/2}B(y)\right)\exp(\sigma x)$$

satisfies the relations

$$\begin{aligned}(\mathcal{L}w)(x,y) &= c_6\gamma^2\varepsilon^{-1/2}B(y)\exp(\sigma x) + \left(-\varepsilon\sigma^2 + \sigma b(x) + d(x)\right) w(x,y) \\ &\geq |(\mathcal{L}v_1)(x,y)| \quad \text{for} \quad (x,y) \in \Omega, \\ w(x,y) &\geq |v_1(x,y)| \quad \text{for} \quad (x,y) \in \Gamma.\end{aligned}$$

Thus, for $\partial_2 u$ Lemma 13 yields the estimate similar to (2.131) for $j = 1$ on $\bar{\Omega}$.

To prove the estimate (2.131) for $j = 2$ we twice differentiate the equation (2.1) with respect to y. Set $v_2 = \partial_{22}u$. Due to (2.2) $v_2 = 0$ on the boundary $\Gamma_{in} \cup \Gamma_{out}$. Since $\partial_{11}u = 0$ on Γ_{tg}, because (2.1), and (2.3) we have

$$|v_2| = |\partial_{22}u| = |-\partial_{11}u + \varepsilon^{-1}(b(x)\partial_1 u + u\partial_1 b - f)| \le c_7\varepsilon^{-1} \text{ on } \Gamma_{tg}. \tag{2.135}$$

As before, the barrier function

$$w(x,y) = c_8\left(1 + \varepsilon^{-1}B(y)\right)\exp(\sigma x)$$

satisfies the assumptions of Lemma 13

$$\begin{aligned}(\mathcal{L}w)(x,y) &\ge |(\mathcal{L}v_2)(x,y)| \quad \text{for} \quad (x,y) \in \Omega,\\ w(x,y) &\ge |v_2(x,y)| \quad \text{for} \quad (x,y) \in \Gamma.\end{aligned}$$

Consequently, for $j = 2$ (2.131) holds on $\bar{\Omega}$.

It remains to prove (2.131) for $j = 3$. We differentiate the equation (2.1) with respect to y three times and set $v_3 = \partial_{222}u$. At first, we estimate the value of $\partial_2 v_3$ on the boundary. From (2.2) we get $\partial_2 v_3 = 0$ on $\Gamma_{in} \cup \Gamma_{out}$. Then differentiating the equation (2.1) with respect to y twice, we use the inequality similar to (2.135). In view of (2.131) for $j = 2$ we have

$$|v_2| \le c_9\varepsilon^{-2} \quad \text{on} \quad \Gamma_{tg}.$$

The barrier function

$$w(x,y) = c_{10}\left(1 + \varepsilon^{-3/2}B(y)\right)\exp(\sigma x)$$

satisfies

$$\begin{aligned}(\mathcal{L}w)(x,y) &\ge |(\mathcal{L}v_3)(x,y)| \quad \text{for} \quad (x,y) \in \Omega,\\ w(x,y) &\ge |v_3(x,y)| \quad \text{for} \quad (x,y) \in \Gamma_{in} \cup \Gamma_{out},\\ \frac{\partial w(x,y)}{\partial n} &\ge \left|\frac{\partial v_3(x,y)}{\partial n}\right| \quad \text{for} \quad (x,y) \in \Gamma_{tg}.\end{aligned}$$

Applying Lemma 14 we complete the proof of the estimate (2.131) for $j = 3$ on $\bar{\Omega}$. The lemma is proved. □

Lemma 25. *Assume that* $\varepsilon > 0$ *is small enough and* (2.3), (2.4) *hold. Then the estimates*

$$\left|\frac{\partial^k u_0}{\partial x^k}\right| \le c_1, \quad k = 0, 1, 2 \tag{2.136}$$

hold.

Proof. Assume that σ is given by (2.10) and ε satisfies (2.12). We derive (2.136) by the comparison principle for the operator $\mathcal{L}_{par}$ (Lemma 23).

Set $d = b'$, then the barrier function

$$w(x,y) = c_2 x \exp(\sigma x) \tag{2.137}$$

with the constant $c_2 = 2\|f\|_\infty$ satisfies the relations

$$\begin{aligned}(\mathcal{L}_{par} w)(x,y) &= c_2 b(x)\exp(\sigma x) + (d(x) + \sigma b(x))\, w(x) \\ &\geq (\mathcal{L}_{par} u_0)(x,y) \quad \text{for} \quad (x,y) \in \Omega, \\ w(x,y) &\geq |u_0(x,y)| \quad \text{for} \quad (x,y) \in \Gamma_{in} \cup \Gamma_{tg}.\end{aligned}$$

Using Lemma 23, we see that u_0 is bounded on $\bar{\Omega}$.

From (2.125) it follows that $\partial_1 u_0 = 0$ on Γ_{tg} and $\partial_{22} u_0 = 0$ on Γ_{in}. Due to this together with (2.3), (2.4), and (2.124) the derivative $\partial_1 u_0$ is bounded on Γ :

$$\partial_1 u_0(x,y) \leq c_3 \quad \text{for} \quad (x,y) \in \Gamma. \tag{2.138}$$

Now we derive (2.136) for $k = 1$. Differentiate the equation (2.124) with respect to x and introduce the notation $v_1 = \partial_1 u_0$. Setting $d = 2b'$ we have

$$(\mathcal{L}_{par} v_1)(x,y) = \partial_1 f - u_0 b'' \quad \text{in} \quad \Omega.$$

In view of (2.138) for v_1, the barrier function w from (2.137) with the constant $c_2 = 2(\|\partial_1 u_0\|_\infty + \|u_0 b''\|_\infty)$ satisfies the assumptions of Lemma 23. Consequently, $\partial_1 u_0$ is bounded on $\bar{\Omega}$.

To prove (2.136) for $k = 2$ we differentiate the equation (2.124) with respect to x twice and introduce the notation $v_2 = \partial_1 u_0$. Setting $d = 3b'$ we get

$$(\mathcal{L}_{par} v_2)(x,y) = \partial_1 f - b'' \partial_1 u_0 - u_0 b''' \quad \text{in} \quad \Omega.$$

Consider this equation on Γ_{in}. Taking into account (2.3) and the fact that u_0 and $\partial_1 u_0$ are bounded, we get the estimate

$$|\partial_{111} u_0| = |\partial_1 v_2| \leq c_4 \quad \text{for} \quad (x,y) \in \Gamma_{in}.$$

From this together with the equality $\partial_{111} u_0 = 0$ on Γ_{tg} it follows that $\partial_1 v_2$ is bounded on the whole boundary Γ. Then the barrier function w with an appropriate constant c_2 satisfies the assumptions of Lemma 23. Hence $\partial_{11} u_0$ is bounded on $\bar{\Omega}$. The proof is complete. $\square$

Lemma 26. *Let* $\varepsilon > 0$ *be sufficiently small and* (2.3), (2.4), (2.20) *be valid for the problem* (2.1)–(2.2). *Then the remainder term in* (2.123) *satisfies*

$$\|\eta\|_\infty \leq c_1, \tag{2.139}$$

$$|\partial_1 \eta(x,y)| \leq c_2(1 + \varepsilon^{-1}\exp(-B_1(1-x)/2\varepsilon)) \quad \textit{on} \quad \bar{\Omega}. \tag{2.140}$$

Proof. First we set $d = b'(x)$ in (2.6) so that $\mathcal{L}u = f$. In view of (2.124), (2.127), and the estimate (2.136) for $j = 2$, by an elementary calculation we show that η in (2.123) satisfies

$$\begin{aligned}(\mathcal{L}\eta)(x,y) &= a_0(x,y) + \frac{1}{\varepsilon}a_1(x,y)A(x,\varepsilon) \\ &+ \frac{1-x}{\varepsilon^2}a_2(x,y)A(x,\varepsilon) \text{ on } \bar{\Omega}\end{aligned} \tag{2.141}$$

where $A(x) = \exp(-(1-x)b(x)/\varepsilon)$ and a_0, a_1, a_2 are bounded functions on $\bar{\Omega}$. In a similar way as in Lemma 19, the right-hand side of (2.141) is evaluated by

$$|L\eta| \le c_3 + \left(c_4\frac{1}{\varepsilon} + c_5\frac{1-x}{\varepsilon^2}\right)A(x,\varepsilon). \tag{2.142}$$

Let us use the barrier function

$$w(x,y) = c_{13}\psi_1(x,y) + c_{14}\psi_2(x,y)$$

where the functions ψ_1, ψ_2 are taken from Lemmata 16, 17 with the constants from Lemma 19. This gives

$$|(\mathcal{L}\eta)(x,y)| \le (\mathcal{L}w)(x,y) \quad \text{in } \Omega.$$

Moreover, we have

$$w \ge 0 = |\eta| \quad \text{on} \quad \Gamma.$$

Thus, all the assumptions of Lemma 16 are satisfied. By this lemma, η is bounded on $\bar{\Omega}$.

Besides, since

$$w(0,y) = w(1,y) = 0 \quad \forall\, y \in [0,1],$$

we have

$$|\partial_1\eta(0,y)| \le c_6 \quad \text{and} \quad |\partial_1\eta(1,y)| \le c_7\varepsilon^{-1}. \tag{2.143}$$

In order to show (2.140), we first differentiate (2.141) with respect to x. Denoting $v = \partial_1\eta$ and setting $d = 2b'$ in (2.6), from (2.136) for $j = 3$ we obtain the representation

$$\begin{aligned}(\mathcal{L}v)(x,y) &= a_3(x,y) \\ &+ \frac{1}{\varepsilon^2}a_4(x,y)A(x,\varepsilon) + \frac{1-x}{\varepsilon^3}a_5(x,y)A(x,\varepsilon)\end{aligned} \tag{2.144}$$

with functions a_3, a_4, and a_5, bounded on $\bar{\Omega}$. The right-hand side of (2.144) can be estimated in the following way:

$$|(\mathcal{L}v)(x,y)| \le c_8 + c_9\varepsilon^{-2}A(x,\varepsilon).$$

Now choosing the barrier function $w = c_{10}\psi_3$ from Lemma 18 with an appropriate constant c_{10}, we get

$$|(\mathcal{L}v)(x,y)| \le (\mathcal{L})w(x,y) \quad \text{in } \Omega,$$

$$w \ge 0 = |v| \quad \text{on} \quad \Gamma_{tg}, \quad w \ge |v| \quad \text{on} \quad \Gamma_{in} \cup \Gamma_{out}.$$

The last inequality follows from (2.143). Thus, due to Lemma 13 we obtain

$$|v| \le w \quad \text{on} \quad \bar{\Omega}$$

that involves (2.140). □

2.3.2 Construction of the fitted quadrature rule.

For the approximation of the regular boundary layer we use the technique considered in the previous section. For the approximation of the parabolic layer we construct the special grid based on the extension method (see [35], [36], [5]).

First we put $h = 1/n$ with even integer $n \ge 2$ and take the uniform grid in the x-direction:

$$x_i := ih, \quad i = 0, 1, ..., n.$$

Next, in the y-direction we algorithmically introduce the graded grid in the y-direction:

$$y_j := \begin{cases} 0, \quad \text{for} \quad j = 0, \\ y_{j-1} + \dfrac{c_0 h}{1 + \varepsilon^{-1/2}\exp(-\gamma y_{j-1}/\sqrt{\varepsilon})}, \quad \text{for} \quad j = 1, ..., n/2, \\ 1 - y_{n-j}, \quad \text{for} \quad j = n/2+1, ..., n. \end{cases} \tag{2.145}$$

The constant c_0 satisfies the condition $y_{n/2} = 1/2$. Unfortunately, this leads to a nonlinear equation in c_0.

We define the mesh size in the y-direction by

$$h_j = y_j - y_{j-1}, \quad j = 1, \dots, n. \tag{2.146}$$

We denote the set of nodes by

$$\bar{\Omega}_h = \{z_{ij} = (x_i, y_j), \quad i, j = 0, 1, \dots, n\},$$

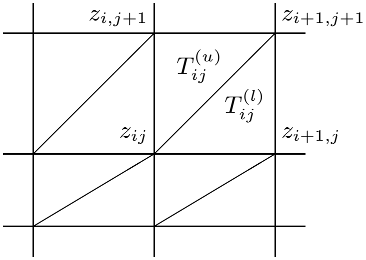

Fig. 4: Fragment the trangulation $\mathcal{T}_h$.

the set of interior nodes by

$$\Omega_h = \{z_{ij} = (x_i, y_j), \quad i, j = 1, 2, \dots, n-1\},$$

and the set of boundary nodes by

$$\Gamma_h = \{z_{ij} = (x_i, y_j),\ i = 0, 1 \text{ and } j = 0, 1, \dots, n; i = 0, 1, \dots, n, \text{ and } j = 0, 1\}.$$

Then the triangulation $\mathcal{T}_h$ is constructed by dividing each *elementary* rectangle $\Omega_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ into two *elementary* triangles by the diagonal passing from (x_i, y_j) to (x_{i+1}, y_{j+1}) (see Fig. 4).

For each interior node $z_{ij} \in \Omega_h$ we introduce the basis function φ_{ij} which equals 1 at the node z_{ij}, equals 0 at any other node of $\bar{\Omega}_h$, and is linear on each elementary triangle of $\mathcal{T}_h$. Denote the linear span of these functions by

$$H^h = \text{span}\{\varphi_{ij}\}_{i,j=1}^{n-1}.$$

With this notations, we formulate the Galerkin problem: *find $u^h \in H^h$ such that*

$$a(u^h, v) = (f, v) \qquad \forall\, v \in H^h. \tag{2.147}$$

As before, in order to ensure the stability and to improve the accuracy of the method, we construct the special quadrature rule that provides good approximation on the smooth and boundary layer components of the solution. Since this technique was described in detail in Section 2.2.2, now we sketch the broad outlines of the construction of the quadrature rule on the nonuniform grid.

Let $T_{ij}^{(l)}$ (or $T_{ij}^{(u)}$, respectively) be an arbitrary elementary triangle of $\mathcal{T}_h$ with the vertices $z_{i,j}$, $z_{i+1,j+1} = (x_{i+1}, y_{j+1})$, and $z_{i+1,j} = (x_{i+1}, y_j)$ (or $z_{i+1,j} = (x_i, y_{j+1})$, respectively) as in Fig. 4. We consider the bilinear form

$$a_T(u, v) = \int_T \left(\left(\varepsilon \frac{\partial u}{\partial x} - bu \right) \frac{\partial v}{\partial x} + \varepsilon \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) d\Omega. \tag{2.148}$$

Its approximation by piecewise linear functions $w^h, v^h \in H^h$, for example, on the triangle $T_{ij}^{(l)}$ has the form

$$a^h_{T^{(l)}_{ij}}(w^h, v^h) = \frac{hh_{j+1}}{2}\Big(\left(\varepsilon b_i(\alpha_{1i} w^h(z_{ij}) + \alpha_{2i} w^h(z_{i+1,j}))\right) \frac{\partial v^h}{\partial x} + \varepsilon \frac{\partial w^h}{\partial y}\frac{\partial v^h}{\partial y}\Big). \tag{2.149}$$

As before, the weights α_{1i} and α_{2i} are chosen in such a way as to satisfy two requirements, namely, to guarantee the first order accuracy for a smooth solution, and to reduce the error of approximation of the difference $a_{T_{ij}}(\rho_0, v^h) - a^h_{T_{ij}}(\rho_0^I, v^h)$ for the regular boundary layer component ρ_0 and its piecewise linear interpolant $\rho_0^I \in H^h$ on each element $T_{ij} \in \mathcal{T}_h$. The first requirement involves the equation

$$\alpha_{1i} + \alpha_{2i} = 1. \tag{2.150}$$

To satisfy the second one, we demand that the equality

$$a_{T_{ij}}(\zeta_i, v^h) = a^h_{T_{ij}}(\zeta_i^I, v^h) \tag{2.151}$$

be valid for the function $\zeta_i(x) = \exp\left(-(1-x)b_i/\varepsilon\right)$ and its piecewise linear interpolant $\zeta_i^I(x,y)$ on T_{ij}. Solving the system (2.150), (2.151), we get the unique solution

$$\alpha_{1i} = \frac{\exp\sigma_i}{(\exp\sigma_i - 1)} - \frac{1}{\sigma_i}, \quad \alpha_{2i} = \frac{1}{\sigma_i} - \frac{1}{\exp\sigma_i - 1} \tag{2.152}$$

where $\sigma_i = b_i h/\varepsilon$. With this weights we obtain the following approximation of the elementary bilinear form (2.149):

$$a^h_{T^{(l)}_{ij}}(w^h, v^h) = \frac{1}{2}hh_{j+1}\Big(\frac{b_i}{\exp\sigma_i - 1}(w^h_{i+1,j} - w^h_{ij}\exp\sigma_i)\frac{\partial v^h}{\partial x} + \varepsilon \frac{\partial w^h}{\partial y}\frac{\partial v^h}{\partial y}\Big). \tag{2.153}$$

In a similar way, on the triangle $T_{ij}^{(u)}$ we have

$$a^h_{T^{(u)}_{ij}}(w^h, v^h) = \frac{1}{2}hh_{j+1}\Big(\frac{b_i}{\exp\sigma_i - 1}(w^h_{i+1,j+1} - w^h_{i,j+1}\exp\sigma_i)\frac{\partial v^h}{\partial x} + \varepsilon \frac{\partial w^h}{\partial y}\frac{\partial v^h}{\partial y}\Big). \tag{2.154}$$

To integrate the right-hand side, we use the simple piecewise constant approximation. This gives the following approximation of the linear form

$$\begin{aligned} f^h_{T^{(l)}_{ij}}(v^h) &= \frac{1}{6} h h_{j+1} (f_{ij} v_{ij} + f_{i+1,j} v_{i+1,j} + f_{i+1,j+1} v_{i+1,j+1}), \\ f^h_{T^{(u)}_{ij}}(v^h) &= \frac{1}{6} h h_{j+1} (f_{ij} v_{ij} + f_{i,j+1} v_{i,j+1} + f_{i+1,j+1} v_{i+1,j+1}). \end{aligned} \tag{2.155}$$

Summing (2.153), (2.154), and (2.155) over all the triangles $T \in \mathcal{T}_h$, we obtain the approximations of the bilinear and linear forms

$$a^h(w^h, v^h) = \sum_{T \in \mathcal{T}_h} a^h_T(w^h, v^h), \tag{2.156}$$

$$f^h(v^h) = \sum_{T \in \mathcal{T}_h} f^h_T(v^h). \tag{2.157}$$

Now we come to the fitted Galerkin problem: *find* $u^h \in H^h$ *such that*

$$a^h(u^h, v^h) = f^h(v^h) \qquad \forall\, v^h \in H^h. \tag{2.158}$$

This problem is equivalent to the system of linear algebraic equations

$$\begin{aligned} &(L^h u^h)_{ij} \equiv \\ &u^h_{ij} \left(\frac{h_j + h_{j+1}}{2} \left(\frac{b_i \exp \sigma_i}{\exp \sigma_i - 1} + \frac{b_{i-1}}{\exp \sigma_{i-1} - 1} \right) + \varepsilon h \left(\frac{1}{h_j} + \frac{1}{h_{j+1}} \right) \right) \\ &- u^h_{i+1,j} \frac{h_j + h_{j+1}}{2} \frac{b_i}{\exp \sigma_i - 1} - u^h_{i-1,j} \frac{h_j + h_{j+1}}{2} \frac{b_{i-1} \exp \sigma_{i-1}}{\exp \sigma_{i-1} - 1} \\ &- u^h_{i,j-1} \varepsilon \frac{h}{h_j} - u^h_{i,j+1} \varepsilon \frac{h}{h_{j+1}} = f_{ij} \frac{h_j + h_{j+1}}{2} h, \quad i, j = 1, 2, ..., n-1; \end{aligned} \tag{2.159}$$

$u^h_{ij} = 0$ for $i = 1, ..., n-1$ and $j = 0, n$ or for $j = 1, ..., n-1$ and $i = 0, n$.

The parameters $\{u^h_{ij}\}_{i,j=1}^{n-1}$ give the solution of the problem (2.158)

$$u^h = \sum_{i,j=1}^{n-1} u_{ij} \varphi_{ij}. \tag{2.160}$$

Eliminate the boundary unknowns and enumerate the remaining unknowns and the equations from 1 to $(n-1)^2$ in the same way (for example, in the lexicographic order). We obtain the shortened system

$$A^h U = F \tag{2.161}$$

where

$$U = (u^h_{1,1}, ..., u^h_{1,n-1}, u^h_{2,1}, ..., u^h_{n-1,n-1})^T,$$
$$F = (f^h(\varphi_{1,1}), ..., f^h(\varphi_{1,n-1}), ..., f^h(\varphi_{n-1,n-1}))^T.$$

Note that the matrix A^h is irreducible [21], has positive diagonal elements and non-positive off-diagonal ones. Then this matrix is diagonal-dominant along columns and strictly diagonal-dominant along some of them. Therefore A^h is an M-matrix. Hence, the system (2.159) satisfies the comparison principle and has unique solution [21].

2.3.3 The properties of the discrete problem

Now we investigate the discrete problem. The following lemma establishes the error of the approximation of the problem (2.1), (2.2) by the grid problem (2.159).

Lemma 27. *Let u be a solution of the problem* (2.1), (2.2) *under the conditions* (2.4), (2.3), (2.70), *and u^h be a solution of the discrete problem* (2.159). *Then the estimate*

$$\begin{aligned} &|\left(L^h(u^h - u^I)\right)_{ij})| \\ &\le c_1 h(h_j + h_{j+1})\left(\varepsilon + h + \exp\left(-(1 - x_{i+1})B_1/2\varepsilon\right)\right) \quad \forall\, i, j = 1, ..., n-1 \end{aligned} \tag{2.162}$$

holds.

Proof. Consider the operator L^h as the sum of two operators of difference differentiation with respect to x and y

$$L^h v = L^h_1 v + L^h_2 v \tag{2.163}$$

where

$$\begin{aligned} \left(L^h_1 v\right)_{ij} = \Bigg(&\left(\frac{b_i \exp \sigma_i}{\exp \sigma_i - 1} + \frac{b_{i-1}}{\exp \sigma_{i-1} - 1}\right) v_{ij} - \frac{b_i}{\exp \sigma_i - 1} v_{i+1,j} \\ &- \frac{b_{i-1} \exp \sigma_{i-1}}{\exp \sigma_{i-1} - 1} v_{i-1,j} \Bigg) \frac{h_j + h_{j+1}}{2}, \end{aligned} \tag{2.164}$$

$$\begin{aligned} \left(L^h_2 v\right)_{ij} &= \varepsilon h \left(\frac{1}{h_j} + \frac{1}{h_{j+1}}\right) v_{ij} - \varepsilon \frac{h}{h_j} v_{i,j-1} - \varepsilon \frac{h}{h_{j+1}} v_{i,j+1}, \\ &i, j = 1, ..., n-1. \end{aligned} \tag{2.165}$$

Using the Tailor expansion at $(x_i, y_j) \in \bar{\Omega}_h$, we have

$$\begin{aligned}\left(L_2^h u\right)_{ij} = \frac{1}{2} &- \varepsilon h(h_j + h_{j+1})\partial_{22}u(x_i, y_j) \\ &- \varepsilon h\left(h_j^2 \pi_1(x_i, y) + h_{j+1}^2 \pi_2(x_i, y)\right).\end{aligned}$$

Because of (2.131) the inequalities

$$\begin{aligned}|\pi_1(x_i, y)| \le c_2\left(1 + \varepsilon^{-3/2}B(y_{j-1})\right),\ |\pi_2(x_i, y)| \le c_3\left(1 + \varepsilon^{-3/2}B(y_j)\right), \\ \text{for } y_j \le 1/2, \\ |\pi_1(x_i, y)| \le c_4\left(1 + \varepsilon^{-3/2}B(y_j)\right),\ |\pi_2(x_i, y)| \le c_5\left(1 + \varepsilon^{-3/2}B(y_{j+1})\right), \\ \text{for } y_j > 1/2\end{aligned}$$

hold. In view of the definition (2.146) of the mesh size in the y direction, we get

$$|h_j \pi_1(x_i, y)|,\ |h_{j+1}\pi_2(x_i, y)| \le c_6 h \varepsilon^{-1}.$$

Then we obtain

$$\left(L_2^h u\right)_{ij} = -\varepsilon h \frac{h_j + h_{j+1}}{2}\partial_{22}u(x_i, y_j) + c_7 h^2(h_j + h_{j+1})\pi_3(x_i, y) \quad (2.166)$$

where $\pi_3(x_i, y)$ is bounded on $[y_{j-1}, y_{j+1}]$.

Using the expansion (2.123), we can write

$$L_1^h u = L_1^h u_0 + L_1^h \rho_0 + \varepsilon L_1^h \eta. \quad (2.167)$$

Now we consider each term in detail.

Using the Taylor expansion of u_0 at (x_i, y_j), we have

$$\begin{aligned}\left(L_1^h u_0\right)_{ij} = \frac{h_j + h_{j+1}}{2}&\Bigg((b_i - b_{i-1})u_{0ij} \\ &+ \left(\frac{b_i}{\exp\sigma_i - 1} - \frac{b_{i-1}\exp\sigma_{i-1}}{\exp\sigma_{i-1} - 1}\right)h\partial_1 u_0(x_i, y_j)\Bigg) \quad (2.168) \\ &+ h^2(h_j + h_{j+1})\pi_4(x, y_j).\end{aligned}$$

Since $b(x)$ is smooth, by the mean value theorem we can write (2.168) in the form

$$\begin{aligned}\left(L_1^h u_0\right)_{ij} = h\frac{h_j + h_{j+1}}{2}&\left(\partial_1 b(x_i, y_j)u_{0ij} + b_i \partial_1 u_0(x_i, y_j)\right) \\ &+ h^2(h_j + h_{j+1})\pi_5(x, y_j)\end{aligned} \quad (2.169)$$

where π_5 is bounded on $[x_{i-1}, x_{i+1}]$.

Further, using the explicit form of the boundary layer function $\rho_0(x,y)$ in (2.164), we obtain

$$\begin{aligned}\left(L_1^h \rho_0\right)_{ij} = \Bigg(\left(\frac{b_i \exp\sigma_i}{\exp\sigma_i - 1} + \frac{b_{i-1}}{\exp\sigma_{i-1} - 1} \right) u_0(1, y_j) s_i \exp\left(-(1 - x_i) b_i/\varepsilon\right) \\ - \frac{b_i}{\exp\sigma_i - 1} u_0(1, y_j) s_{i+1} \exp\left(-(1 - x_{i+1}) b_{i+1}/\varepsilon\right) \\ - \frac{b_{i-1}\exp\sigma_{i-1}}{\exp\sigma_{i-1} - 1} u_0(1, y_j) s_{i-1} \exp\left(-(1 - x_{i-1}) b_{i-1}/\varepsilon\right) \Bigg) \frac{h_j + h_{j+1}}{2}.\end{aligned} \tag{2.170}$$

Using the mean value theorem, the smoothness of $b(x)$, and (2.33), we have the estimate

$$\begin{aligned}&|\exp\left(-(1 - x_{i+1}) b_{i+1}/\varepsilon\right) - \exp\left(-(1 - x_{i+1}) b_i/\varepsilon\right)| \\ &\le |b_i - b_{i-1}| \frac{1 - x_{i-1}}{\varepsilon} \exp\left(-(1 - x_{i-1}) b_i^*/\varepsilon\right) \le c_8 h \exp\left(-(1 - x_{i+1}) B_1/2\varepsilon\right)\end{aligned}$$

where $b_i \in [B_1, B_2]$. Rearranging the terms in (2.170) and taking into consideration the smoothness of $s(x)$, we get

$$\begin{aligned}\left(L_1^h \rho_0\right)_{ij} = \Bigg(\Bigg(\frac{b_i \exp\sigma_i}{\exp\sigma_i - 1} \exp\left(-(1 - x_i) b_i/\varepsilon\right) \\ - \frac{b_i}{\exp\sigma_i - 1} \exp\left(-(1 - x_{i+1}) b_i/\varepsilon\right) \Bigg) \\ + \Bigg(\frac{b_{i-1}}{\exp\sigma_{i-1} - 1} \exp\left(-(1 - x_i) b_{i-1}/\varepsilon\right) \\ - \frac{b_{i-1}\exp\sigma_{i-1}}{\exp\sigma_{i-1} - 1} \exp\left(-(1 - x_i) b_i/\varepsilon\right) \Bigg) \\ + h\pi_6(x, y_j) \Bigg) \frac{h_j + h_{j+1}}{2} u_0(1, y_j) = h(h_j + h_{j+1}) \pi_6(x, y_j)\end{aligned} \tag{2.171}$$

where

$$|\pi_6(x, y_j)| \le c_9 \exp\left(-(1 - x_{i+1}) B_1/2\varepsilon\right), \quad x \in [x_{i-1}, x_{i+1}].$$

By the mean value theorem, from (2.164) the equality

$$\varepsilon \left(L_1^h \eta\right)_{ij} = \frac{1}{2} \varepsilon (h_j + h_{j+1})(b_i - b_{i-1}) \eta_{ij} + h(h_j + h_{j+1}) \pi_7(x, y_j) \tag{2.172}$$

follows where due to (2.140) we have

$$|\pi_7(x,y_j)| \le c_{10}\left(\varepsilon + \exp\left(-(1-x_{i+1})B_1/2\varepsilon\right)\right), \quad x \in [x_{i-1}, x_{i+1}]. \tag{2.173}$$

Taking into consideration (2.139) and the smoothness of $b(x)$, the equality (2.172) can be rewritten as

$$\varepsilon\left(L_1^h \eta\right)_{ij} = h(h_j + h_{j+1})\pi_8(x,y_j) \tag{2.174}$$

with the estimate of π_8 similar to (2.173).

Thus, combining (2.166), (2.169), (2.171), and (2.174), we obtain

$$\begin{aligned}\left|\left(L^h(u^h-u)\right)_{ij}\right| = \Big|\left(L^h u^h\right)_{ij} + \Big(&\varepsilon\partial_{22}u(x_i,y_j) - b_i\partial_1 u_0(x_i,y_j) \\ &- u_{0ij}\partial_1 b(x_i)\Big)h\frac{h_j+h_{j+1}}{2} + h(h_j+h_{j+1})\pi_{10}(x,y)\Big|\end{aligned} \tag{2.175}$$

where

$$\begin{aligned}|\pi_{10}(x,y)| \le c_{10}\left(h + \varepsilon + \exp\left(-(1-x_{i+1})B_1/2\varepsilon\right)\right),& \\ x \in [x_{i-1}, x_{i+1}],\ y \in [x_{j-1}, y_{j+1}].&\end{aligned}$$

At the nodes of the grid the following equality holds:

$$\left(L^h u^h\right)_{ij} = h\frac{h_j+h_{j+1}}{2} f_{ij} - (Lu)_{ij}.$$

Now we consider the expansion of the differential operator L similar to (2.163)

$$Lv = L_1 v + L_2 v$$

where

$$L_1 v = -\varepsilon\partial_{11}v + \partial_1(b(x)v), \quad L_2 v = -\varepsilon\partial_{22}v.$$

Use the expansion (2.123) and write

$$L_1 u = L_1 u_0 + L_1\rho_0 + \varepsilon L_1\eta.$$

Because of (2.33), for $L_1\rho_0$ the following estimate holds:

$$\begin{aligned}L_1\rho_0 &= g(y)\exp\left(-(1-x)b(x)/\varepsilon\right)\Big((1-x)\partial_{11}b \\ &\quad + \varepsilon^{-1}(1-x)b(x)\partial_1 b - \partial_1 b + \varepsilon^{-1}(1-x)^2\left(\partial_1 b\right)^2\Big)h\frac{h_j+h_{j+1}}{2} \\ &\le c_{11}h\frac{h_j+h_{j+1}}{2}\exp\left(-(1-x)b(x)/2\varepsilon\right) \quad \text{for} \quad x \in [x_i, x_{i+1}].\end{aligned}$$

Taking into account (2.139) and (2.140), we get

$$\varepsilon L_1 \eta \le c_{12} h \frac{h_j + h_{j+1}}{2} \big(\varepsilon + \exp\left(-(1-x)b(x)/2\varepsilon\right)\big) \quad \text{for} \quad x \in [x_i, x_{i+1}].$$

Thus, with (2.136) for $\partial_{11} u_0$, we obtain

$$(Lu)_{ij} = h \frac{h_j + h_{j+1}}{2} \left(b_i \partial_1 u_0(x_i, y_j) + u_{0ij} \partial_1 b(x_i) + \pi_{11}(x,y)\right) \tag{2.176}$$

where π_{11} is estimated similarly to (2.173).

By substituting (2.176) into (2.175) we complete the proof. □

In order to prove the convergence of the numerical solution to the exact one, we define the barrier function for the right-hand side of (2.162).

Lemma 28. *There exist grid functions φ^h and ψ^h on $\bar{\Omega}_h$ with the properties*

$$|\varphi^h| \le c_1 \quad \text{on} \quad \Omega_h, \tag{2.177}$$

$$|\psi^h| \le c_2 h \quad \text{on} \quad \Omega_h, \tag{2.178}$$

such that

$$L^h \varphi^h \ge h h_{j+1} \quad \text{in} \quad \Omega_h, \tag{2.179}$$

$$\varphi^h \ge 0 \quad \text{on} \quad \Gamma_h; \tag{2.180}$$

$$L^h \psi^h \ge h h_{j+1} \exp\left(-B_1(1 - x_{i+1})/2\varepsilon\right) \quad \text{in} \quad \Omega_h, \tag{2.181}$$

$$\psi^h \ge 0 \quad \text{on} \quad \Gamma_h. \tag{2.182}$$

This lemma is proved in much the same way as Lemma 21.

Finally, the following convergence result is valid.

Theorem 29. *Assume that* (2.4), (2.3) *hold. Then there exist constants h_0 and c independent of h and ε such that $\forall\, h \le h_0$ and for $\varepsilon \le h$ the solution u^h of the problem* (2.159) *satisfies the estimate*

$$\max_{\hat{\Omega}_h} |u - u^h| \le ch \tag{2.183}$$

where u is the solution of the problem (2.18), (2.19).

The proof follows from Lemmata 27 and 28 as in the previous case.

Thus, we constructed the grid problem for the convection-diffusion problem with regular and parabolic boundary layers. Its solution converges to the exact one with the first order in the uniform discrete norm. The numerical experiments described in Chapter 3 confirm this.

3 Numerical solution of the discrete problem

3.1 Numerical experiments in the one-dimensional case

As a test example we considered the problem

$$-\varepsilon u'' + ((1+2x)u)' = f, \quad x \in (0,1),$$
$$u(0) = u(1) = 0$$

where

$$f(x) = 6x^2 + 2x - 2\varepsilon + 2d, \quad d = \frac{\exp(-2/\varepsilon)}{1-\exp(-2/\varepsilon)}.$$

The parameter ε was taken in the range from 1/10 to 1/5120. The exact solution of this problem is given by

$$u(x) = x^2 + d - (d+1)\exp\left(\frac{x^2+x-2}{\varepsilon}\right).$$

We compared the numerical results obtained by the stable upwind scheme

$$\begin{gathered} -\frac{\varepsilon}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + b_i \frac{u_i - u_{i-1}}{h} + b_i' u_i = f_i, \\ i = 1, ..., n-1, \quad u_0 = u_n = 0, \end{gathered} \tag{3.1}$$

by the difference scheme with exponential fitting (see [23])

$$\begin{gathered} -\frac{\varepsilon\sigma_i}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{b_i}{2h}(u_{i+1} - u_{i-1}) + b_i' u_i = f_i, \\ u_0 = u_n = 0 \end{gathered} \tag{3.2}$$

with the variable fitting coefficient $\sigma_i = \dfrac{b_i h}{2\varepsilon}\mathrm{cth}\left(\dfrac{b_i h}{2\varepsilon}\right)$; by the proposed two schemes (1.53)–(1.54) and (1.80)–(1.81); and by the first-order scheme from [121]. The number n of the nodes of the grid varied from 10 to 320 and the mesh size was difined as $h = 1/n$. The error of the numerical solution was calculated exactly:

$$\delta(n) = \|u - u^h\|_{\infty,h} = \max_{\omega^h} |u_{ij} - u_{ij}^h|.$$

Table 1: The error of the simple upwind scheme (3.1).

ε	h					
	1/10	1/20	1/40	1/80	1/160	1/320
1/10	1.51_{10}-1	1.53_{10}-1	8.98_{10}-2	5.33_{10}-2	2.86_{10}-2	1.48_{10}-2
1/20	8.96_{10}-2	1.74_{10}-1	1.65_{10}-1	9.64_{10}-2	5.66_{10}-2	3.02_{10}-2
1/40	4.76_{10}-2	1.13_{10}-1	1.87_{10}-1	1.71_{10}-1	9.98_{10}-2	5.83_{10}-2
1/80	5.23_{10}-2	4.79_{10}-2	1.26_{10}-1	1.93_{10}-1	1.74_{10}-1	1.02_{10}-1
1/160	5.37_{10}-2	2.87_{10}-2	6.20_{10}-2	1.33_{10}-1	1.97_{10}-1	1.75_{10}-1
1/320	5.41_{10}-2	2.44_{10}-2	2.44_{10}-2	6.93_{10}-2	1.37_{10}-1	1.99_{10}-1
1/640	5.40_{10}-2	3.01_{10}-2	1.55_{10}-2	3.21_{10}-2	7.30_{10}-2	1.39_{10}-1
1/1280	5.72_{10}-2	3.02_{10}-2	1.58_{10}-2	1.23_{10}-2	3.60_{10}-2	7.50_{10}-2
1/2560	5.80_{10}-2	3.02_{10}-2	1.59_{10}-2	8.05_{10}-3	1.64_{10}-2	3.80_{10}-2
1/5120	5.93_{10}-2	3.03_{10}-2	1.59_{10}-2	8.11_{10}-3	6.21_{10}-3	1.84_{10}-2

Table 2: The error of the fitted first-order finite element scheme from [128].

ε	h					
	1/10	1/20	1/40	1/80	1/160	1/320
1/10	1.71_{10}-2	3.01_{10}-3	3.83_{10}-3	2.54_{10}-3	1.43_{10}-3	7.55_{10}-4
1/20	3.49_{10}-2	8.04_{10}-3	1.90_{10}-3	2.24_{10}-3	1.46_{10}-3	8.16_{10}-4
1/40	4.95_{10}-2	1.73_{10}-2	3.88_{10}-3	1.05_{10}-3	1.21_{10}-3	7.82_{10}-4
1/80	5.70_{10}-2	2.49_{10}-2	8.62_{10}-3	1.90_{10}-3	5.51_{10}-4	6.27_{10}-4
1/160	6.07_{10}-2	2.89_{10}-2	1.25_{10}-2	4.30_{10}-3	9.43_{10}-4	2.82_{10}-4
1/320	6.26_{10}-2	3.09_{10}-2	1.45_{10}-2	6.25_{10}-3	2.14_{10}-3	4.69_{10}-4
1/640	6.36_{10}-2	3.19_{10}-2	1.55_{10}-2	7.28_{10}-3	3.12_{10}-3	1.07_{10}-3
1/1280	6.40_{10}-2	3.24_{10}-2	1.60_{10}-2	7.79_{10}-3	3.64_{10}-3	1.56_{10}-3
1/2560	6.43_{10}-2	3.26_{10}-2	1.63_{10}-2	8.05_{10}-3	3.90_{10}-3	1.82_{10}-3
1/5120	6.44_{10}-2	3.27_{10}-2	1.64_{10}-2	8.18_{10}-3	4.03_{10}-3	1.95_{10}-3

The numerical results are given in Tables 1–5 and in Figures 5–7. In the figures the error of the simple upwind scheme is marked by the number 2, the error of the first-order scheme from [121] with the fitted quadrature rule is marked by 3, the error of the difference scheme with exponential fitting is marked by 4, and the error of the presented finite element scheme with the nonlinear quadrature rule is marked by 5. For comparison we show the straight lines with slopes $\text{tg}\varphi = 1$ and $\text{tg}\varphi = 2$ which are marked by 1 and 6 respectively. The numerical results for the presented scheme with the linear qudrature rule does not differ visually from the polygonal line 5 and are not shown in the figures.

Table 3: The error of the difference scheme (3.2) with exponential fitting.

ε	h					
	1/10	1/20	1/40	1/80	1/160	1/320
1/10	1.58_{10}-2	6.23_{10}-3	2.79_{10}-3	6.37_{10}-4	1.08_{10}-4	1.77_{10}-5
1/20	3.07_{10}-2	9.29_{10}-3	2.93_{10}-3	1.37_{10}-3	3.15_{10}-4	5.31_{10}-5
1/40	4.51_{10}-2	1.65_{10}-2	5.02_{10}-3	1.41_{10}-3	6.80_{10}-4	1.56_{10}-4
1/80	5.25_{10}-2	2.38_{10}-2	8.69_{10}-3	2.63_{10}-3	7.01_{10}-4	3.38_{10}-4
1/160	5.63_{10}-2	2.77_{10}-2	1.22_{10}-2	4.47_{10}-3	1.35_{10}-3	3.58_{10}-4
1/320	5.81_{10}-2	2.97_{10}-2	1.42_{10}-2	6.19_{10}-3	2.27_{10}-3	6.82_{10}-4
1/640	5.91_{10}-2	3.07_{10}-2	1.52_{10}-2	7.20_{10}-3	3.12_{10}-3	1.14_{10}-3
1/1280	5.95_{10}-2	3.12_{10}-2	1.57_{10}-2	7.71_{10}-3	3.62_{10}-3	1.56_{10}-3
1/2560	5.98_{10}-2	3.14_{10}-2	1.60_{10}-2	7.97_{10}-3	3.88_{10}-3	1.82_{10}-3
1/5120	5.99_{10}-2	3.15_{10}-2	1.61_{10}-2	8.10_{10}-3	4.01_{10}-3	1.95_{10}-3

Table 4: The error of the fitted finite element scheme (1.53)–(1.54) with the linear quadrature rule.

ε	h					
	1/10	1/20	1/40	1/80	1/160	1/320
1/10	2.53_{10}-3	1.19_{10}-3	3.74_{10}-4	1.02_{10}-4	2.66_{10}-5	6.79_{10}-6
1/20	1.29_{10}-3	8.30_{10}-4	3.80_{10}-4	1.18_{10}-4	3.18_{10}-5	8.25_{10}-6
1/40	1.51_{10}-3	6.55_{10}-4	2.94_{10}-4	1.32_{10}-4	4.95_{10}-5	1.20_{10}-5
1/80	2.29_{10}-3	3.98_{10}-4	3.29_{10}-4	1.15_{10}-4	5.09_{10}-5	2.75_{10}-5
1/160	2.77_{10}-3	6.00_{10}-4	1.02_{10}-4	1.65_{10}-4	4.95_{10}-5	2.74_{10}-5
1/320	3.00_{10}-3	7.01_{10}-4	1.53_{10}-4	3.21_{10}-5	8.25_{10}-6	2.27_{10}-5
1/640	3.11_{10}-3	7.63_{10}-4	1.79_{10}-4	3.87_{10}-5	2.24_{10}-5	4.12_{10}-5
1/1280	3.16_{10}-3	7.92_{10}-4	1.92_{10}-4	4.52_{10}-5	9.72_{10}-6	1.28_{10}-5
1/2560	3.19_{10}-3	8.06_{10}-4	2.00_{10}-4	4.84_{10}-5	1.13_{10}-5	2.44_{10}-5
1/5120	3.20_{10}-3	8.12_{10}-4	2.03_{10}-4	5.00_{10}-5	1.22_{10}-5	2.84_{10}-6

The study of the behaviour of the error of the simple upwind scheme (3.1) shows that for $\varepsilon > h$ (Fig. 5) this scheme has the first-order accuracy, but with ε decreasing the order of accuracy decreases. In [102] it was shown that the scheme (3.2) with exponential fitting has the second-order accuracy for $\varepsilon > h$ (Fig. 5) and only the first-order accuracy for small value of ε. This is seen in Figs. 5–7, where the slope of the polygonal line 4 is changed at around $\varepsilon = h$. Calculations for the scheme from [121] confirm the first-order convergence for small values of the diffusion parameter. The presented schemes have the second-order convergence not only for $\varepsilon < h/2$ that the theoretically proved but for $\varepsilon > 2h$ as well. The results (see Tables 4, 5 and

Table 5: The error of the fitted finite element scheme (1.80)–(1.81) with the nonlinear quadrature rule.

ε	h					
	1/10	1/20	1/40	1/80	1/160	1/320
1/10	2.27_{10}-3	1.50_{10}-3	4.11_{10}-4	1.12_{10}-4	2.86_{10}-5	7.22_{10}-6
1/20	9.24_{10}-4	1.12_{10}-3	7.13_{10}-4	1.94_{10}-4	5.20_{10}-5	1.32_{10}-5
1/40	1.84_{10}-3	2.39_{10}-4	5.54_{10}-4	3.47_{10}-4	9.43_{10}-5	2.49_{10}-5
1/80	2.46_{10}-3	4.67_{10}-4	6.02_{10}-5	2.75_{10}-4	1.71_{10}-4	4.64_{10}-5
1/160	2.82_{10}-3	6.29_{10}-4	1.17_{10}-4	2.75_{10}-5	1.37_{10}-4	8.47_{10}-5
1/320	3.01_{10}-3	7.20_{10}-4	1.59_{10}-4	2.94_{10}-5	1.73_{10}-5	6.83_{10}-5
1/640	3.11_{10}-3	7.69_{10}-4	1.81_{10}-4	3.99_{10}-5	7.38_{10}-6	9.53_{10}-6
1/1280	3.16_{10}-3	7.94_{10}-4	1.94_{10}-4	4.57_{10}-5	9.99_{10}-6	1.85_{10}-6
1/2560	3.19_{10}-3	8.06_{10}-4	2.00_{10}-4	4.84_{10}-5	1.14_{10}-5	2.50_{10}-6
1/5120	3.20_{10}-3	8.13_{10}-4	2.03_{10}-4	5.03_{10}-5	1.22_{10}-5	2.86_{10}-6

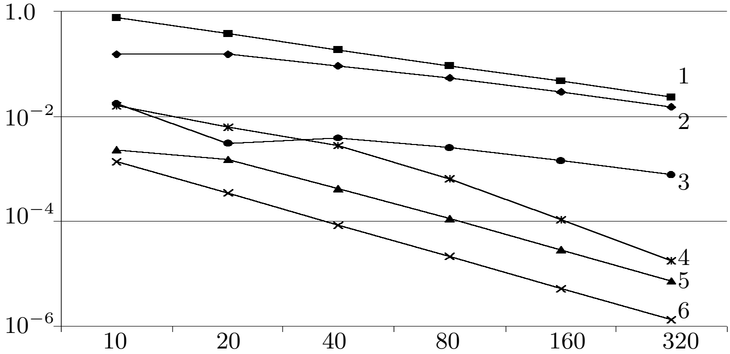

Fig. 5: The maximum error $\delta(n)$ in the one-dimensional case for $\varepsilon = 1/10$.

Figs. 5–7, polygonal line 5) show that for all values of ε these scheme are more accurate than those consider here.

3.2 Test example in the two-dimensional case

Let Ω be the square $(0,1) \times (0,1)$ with the boundary Γ. As a test example we consider the problem

$$-\varepsilon \Delta u + \partial_1 u = 1 \quad \text{in} \quad \Omega, \tag{3.3}$$

$$u = 0 \quad \text{on} \quad \Gamma. \tag{3.4}$$

The solution of this problem has a parabolic boundary layer along the boundary Γ_{tg} and a regular boundary layer near Γ_{out}.

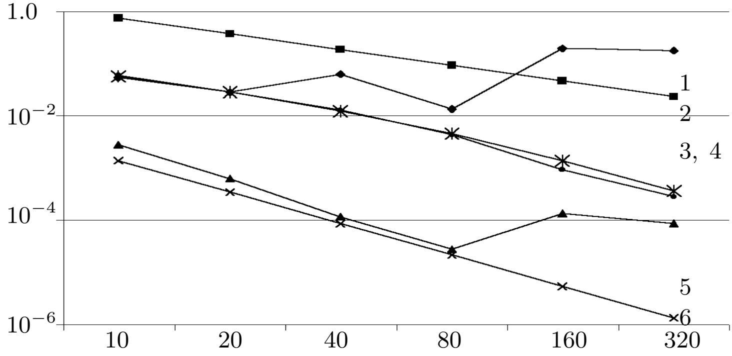

Fig. 6: The maximum error $\delta(n)$ in the one-dimensional case, for $\varepsilon = 1/160$.

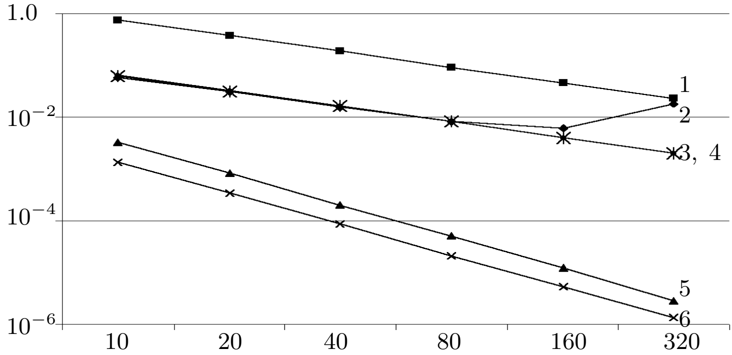

Fig. 7: The maximum error $\delta(n)$ in the one-dimensional case for $\varepsilon = 1/5120$.

The calculations were done on grids uniform in the x-direction. To refine the grid in the y-direction in the parabolic boundary layer, two approaches were considered. The first approach has been proposed by N.S.Bakhvalov in [5]. This approach uses the estimates of the normal derivative of the solution. We consider two kinds of these grids. The second approach has been considered by G.I.Shishkin ([111], [57]). He use the grid with a piecewise constant mesh size that is refined in the parabolic boundary layer.

To solve the discrete problem we applied the pointwise and block Gauss-Seidel methods. We also use the cascadic multigrid algorithm where the interpolation on a coarser grid is taken as the initial guess on a finer one.

3.3 The grids

First we construct the grid in the y-direction according to the works by N.S.Bakhvalov [5] and V.D.Liseikin [35], [36].

Define the nodes of the grid on the segment $[0,1]$ by a non-singular transformation $\lambda(q) : [0,1] \to [0,1]$ in the following way:

$$y_j = \lambda(jh), \quad j = 0,1,...,N, \quad h = 1/N. \tag{3.5}$$

The generating function $\lambda(q)$ is taken so that the difference of the values of the solution at the neighboring nodes in the y-direction is uniformly founded:

$$|u(x, y_{j+1}) - u(x, y_j)| \le c_1 h, \quad j = 0,1,...,N-1.$$

This condition is satisfied if $\lambda(q)$ is a piecewise smooth function and

$$\left| \frac{\partial u(x, \lambda(q))}{\partial q} \right| \le c_2, \quad q \in (0,1). \tag{3.6}$$

According to [35], the use of the estimates (2.131) of the derivative $\partial_2 u(x,y)$ instead of the derivative itself leads to the stronger condition

$$\left| \frac{\partial^k u(x, \lambda(q))}{\partial q^k} \right| \le c_3, \quad 0 \le q \le 1, \quad k > 1. \tag{3.7}$$

Since the solution has two parabolic boundary layers in Ω near the boundaries $\Gamma_{tg}^0 = \{(x,y) : x \in [0,1], \quad y = 0\}$ and $\Gamma_{tg}^1 = \{(x,y) : x \in [0,1], \quad y = 1\}$, we consider the function $\lambda(q)$ which is symmetric about the point $q = 0.5$. The explicit form of the local transformation $\lambda(q)$ in the vicinity of a parabolic boundary layer, for example, near Γ_{tg}^0, can be found as the solution of the problem

$$\frac{dq}{dy} = c_4 \exp\left(-\gamma y/\sqrt{\varepsilon}\right), \quad q(0) = 0, \quad c_4 = 1/\int_0^{q_*} \exp\left(-\gamma t/\sqrt{\varepsilon}\right) dt$$

where $q_* > q$ is the thickness of a boundary layer.

Then on $[0, q_*]$ the generating function has the form

$$\lambda(q) = \sqrt{\varepsilon} \ln\left(1 + 4(1 - \sqrt{\varepsilon})q\right), \quad 0 \le q \le q_*. \tag{3.8}$$

On the segment $[q_*, 0.5]$ the function $\lambda(q)$ is the tangent $\gamma q + \delta$ to the curve (3.8) at the point q_*. The point q_* of sewing and the parameters γ, δ are obtained by the following iterative process:

1. the point $q_*^0 = h[n/4]$ is taken as an initial guess;
2. with the value q_*^k we construct the straight line $\gamma^k q+\delta^k$ passing through the points $(q_*^k, \lambda(q_*^k))$ and $(0.5,\ 0.5)$;
3. we determine q_*^{k+1} from the equation

$$\frac{\partial\lambda(q_*^{k+1})}{\partial q} = \gamma^k;$$

4. if $|q_*^{k+1} - q_*^k| > \delta_{step}$ with the a priori chosen error δ_{step} then go to step (2) else q_*^k is chosen as the point of sewing, γ^k and δ^k are taken as the parameters of the straight line, and the iterative process is terminated.

Thus, the generating function for the Bakhvalov grid has the form

$$\lambda(q) = \begin{cases} \sqrt{\varepsilon}\ln\left(1 + 4(1-\sqrt{\varepsilon})q\right), & 0 \le q \le q_*, \\ \gamma q + \delta, & q_* \le q \le 0.5, \\ 1 - \lambda(1-q), & 0.5 \le q \le 1. \end{cases}$$

We can consider the grids presented in Chapter 2 as grids of the Bakhvalov type, since they are constructed using the information on the behaviour of the normal derivative of the solution.

In this case unlike (3.5) the generating function can not be defined exactly. The nodes of the grid are determined by the equalities

$$\begin{aligned} &y_0 = 0, \\ &y_j = y_{j-1} + \frac{c_0 h}{1 + \varepsilon^{-1/2}\exp\left(-\gamma y_{j-1}/\sqrt{\varepsilon}\right)}, \quad j = 1, 2, ..., \frac{n}{2}, \\ &y_{n/2} = 0.5, \\ &y_j = 1 - y_{n-j}, \quad j = \frac{n}{2} + 1, ..., n. \end{aligned} \tag{3.9}$$

Here $h = 1/n$. The grid parameter c_0 is determined from the nonlinear equation

$$y_{n/2} = y_{n/2-1} + \frac{c_0 h}{1 + \varepsilon^{-1/2}\exp\left(-\gamma y_{n/2-1}/\sqrt{\varepsilon}\right)}.$$

In the numerical experiments we used the Jacobi-type iterative process.

Another way of grid refinement that we used in the numerical experiments has been proposed by G.I.Shishkin (see, for example, [111], [57]).

Let $n+1$ be the number of nodes in the y-direction. The thickness of the numerical parabolic boundary layer is determined by

$$\tau = \min\{1/4, \sqrt{\varepsilon}\ln n\}.$$

The mesh-size is piecewise constant. In the vicinity of the parabolic boundary layer $y \in [0, \tau] \cup [1-\tau, 1]$ it is taken by

$$h_p = \frac{\tau}{[n/4]}$$

and in the remaining part $y \in [\tau, 1-\tau]$ it is determined by

$$h_r = \frac{1-2\tau}{[n/2]}.$$

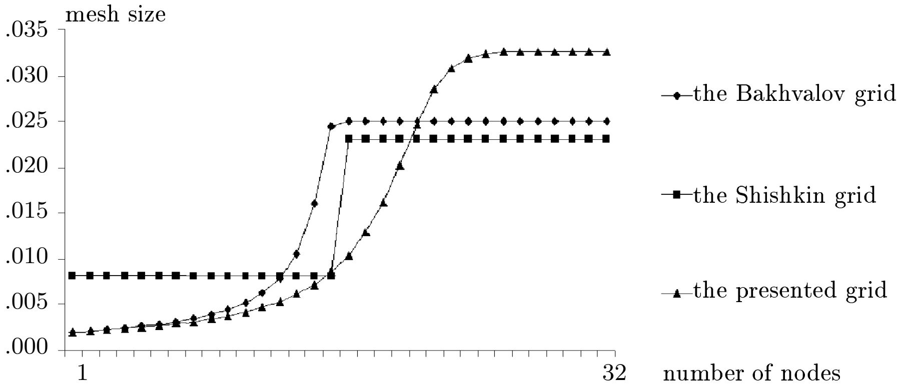

Fig. 8: The mesh-size functions.

The mesh size functions for each grid are demonstrated in Fig. 8. The number of nodes in the vicinity of the parabolic boundary layer and the thickness of the layer for the Bakhvalov and Shishkin grids given in Table 6 for various values n for $\varepsilon = 10^{-3}$. On the presented grid (3.9) the thickness of the boundary layer is not clearly defined.

3.4 Methods for solving the discrete problem

In Chapter 2 the discrete problem (2.161) was obtained. To solve it we apply the pointwise and block Gauss-Seidel methods. Now we briefly describe these methods according to [46].

We represent the matrix A^h in (2.161) as the sum of the lower triangular matrix B^h with a nonzero diagonal and the upper triangular matrix C^h with the zero diagonal

$$A^h = B^h + C^h \tag{3.10}$$

Table 6: The characteristics of the grids.

n	number of nodes in bound. layer		thickness of bound. layer	
	Bakhvalov	Shishkin	Bakhvalov	Shishkin
32	8	8	5.943_{10}-2	1.096_{10}-1
64	16	16	7.540_{10}-2	1.315_{10}-1
128	31	32	7.540_{10}-2	1.554_{10}-1
256	62	64	8.107_{10}-2	1.750_{10}-1
512	123	128	8.107_{10}-2	1.972_{10}-1
1024	245	256	8.107_{10}-2	2.191_{10}-1
2048	489	512	8.107_{10}-2	2.411_{10}-1

where

$$B^h = \begin{pmatrix} a^h_{11} & 0 & 0 & \cdots & 0 & 0 \\ a^h_{21} & a^h_{22} & 0 & \cdots & 0 & 0 \\ a^h_{31} & a_{32} & a^h_{33} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a^h_{M1} & a^h_{M2} & a^h_{M3} & \cdots & a^h_{MM-1} & a^h_{MM} \end{pmatrix},$$

$$C^h = \begin{pmatrix} 0 & a^h_{12} & a^h_{13} & \cdots & a^h_{1M-1} & a^h_{1M} \\ 0 & 0 & a^h_{23} & \cdots & a^h_{2M-1} & a^h_{2M} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & a^h_{M-1,M} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$M = (n-1) \times (n-1)$.

Using these notations we rewrite the Gauss-Seidel method as

$$B^h u^{(k+1)} + C^h u^{(k)} = F, \; k = 0, 1, ...; \quad u^{(0)} = 0. \tag{3.11}$$

From here on, k is the number of iteration steps.

The iterative process (3.11) can be rewritten in the canonical form

$$B^h \left(u^{(k+1)} - u^{(k)} \right) + A^h u^{(k)} = F. \tag{3.12}$$

The operator B^h is a triangular matrix, hence it is not self-adjoint.

Taking into account (2.159), we can write the system (2.161) in the form

$$-a_{ij}u_{i-1,j} + b_{ij}u_{ij} - c_{ij}u_{i+1,j} - d_{ij}u_{i,j-1} - e_{ij}u_{i,j+1} = f_{ij}, \qquad i,j = 1,...,n-1 \tag{3.13}$$

where

$$\begin{aligned}
a_{ij} &= \frac{h_j + h_{j+1}}{2}\,\frac{b_{i-1}\exp\sigma_{i-1}}{\exp\sigma_{i-1} - 1}, \qquad d_{ij} = \varepsilon\frac{h}{h_j},\\
c_{ij} &= \frac{h_j + h_{j+1}}{2}\,\frac{b_i}{\exp\sigma_i - 1}, \qquad e_{ij} = \varepsilon\frac{h}{h_{j+1}},\\
b_{ij} &= a_{i+1,j} + c_{i-1,j} + d_{ij} + e_{ij}.
\end{aligned} \tag{3.14}$$

Then the pointwise Gauss-Seidel method (3.11) can be rewritten in the form

$$u_{ij}^{(k+1)} = \frac{1}{b_{ij}}\left(f_{ij} + a_{ij}u_{i-1,j}^{(k+1)} + c_{ij}u_{i+1,j}^{(k)} + d_{ij}u_{i,j-1}^{(k+1)} + e_{ij}u_{i,j+1}^{(k)}\right), \tag{3.15}$$

$$i,j = 1,...,n-1, \quad k = 0,1,\dots .$$

The numerical experiment demonstrated that this method failes, especially on the Bakhvalov grids. This method is sensitive to the grid refinement in a parabolic boundary layer. We can see this in the example given below.

At the same time A^h has a certain block structure. We use this property and consider the block Gauss-Seidel method. We denote by $\mathbf{U}_i = (u_{i,1}, u_{i,2}, ..., u_{i,n-1})^T$ the vector whose components are the values u_{ij} of the grid function for fixed i. Then the grid equations (3.13) can be rewritten as the system of three-level vector equations

$$-A_i\mathbf{U}_{i-1} + B_i\mathbf{U}_i - C_i\mathbf{U}_{i+1} = \mathbf{F}_i, \quad j = 1,2,...,n-1, \tag{3.16}$$

where A_i and C_i are diagonal $(n-1)\times(n-1)$ matrices. Here

$$diag\,(A_i) = (a_{i,1}, a_{i,2}, ..., a_{i,n-1})^T,$$

$$diag\,(C_i) = (c_{i,1}, c_{i,2}, ..., c_{i,n-1})^T,$$

$$\mathbf{F}_i = (f_{i,1}, f_{i,2}, ..., f_{i,n-1})^T,$$

and B_i is a tridiagonal $(n-1)\times(n-1)$ matrix

$$B_i = \begin{pmatrix}
b_{i1} & -e_{i1} & 0 & 0 & \cdots & 0 & 0 & 0\\
-d_{i2} & b_{i2} & -e_{i2} & 0 & \cdots & 0 & 0 & 0\\
0 & -d_{i3} & b_{i3} & -e_{i3} & \cdots & 0 & 0 & 0\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
0 & 0 & 0 & 0 & \cdots & -d_{i,n-2} & b_{i,n-2} & -e_{i,n-2}\\
0 & 0 & 0 & 0 & \cdots & 0 & b_{i,n-1} & -e_{i,n-1}
\end{pmatrix}.$$

The block Gauss-Seidel method for the system (3.16) has the form

$$B_i\mathbf{U}_i^{(k+1)} = \mathbf{F}_i + A_i\mathbf{U}_{i-1}^{(k+1)} + C_i\mathbf{U}_{i+1}^{(k)}, \qquad i = 1, 2, ..., n-1, \quad k = 0, 1.... \tag{3.17}$$

To determin $\mathbf{U}_i^{(k+1)}$, one have to invert the tridiagonal matrix B_i. To do this, the sweep method can be applied.

The pointwise representation of (3.17) has the form

$$-d_{ij}u_{i,j-1}^{(k+1)} + b_{ij}u_{ij}^{(k+1)} - e_{ij}u_{i,j+1}^{(k+1)} = f_{ij} + a_{ij}u_{i-1,j}^{(k+1)} + c_{ij}u_{i+1,j}^{(k)}, \tag{3.18}$$
$$i = 1, ..., n-1, \quad j = 1, ..., n-1, \quad k = 0, 1....$$

The numerical experiments showed the advantage of the block Gauss-Seidel method over the pointwise one.

Moreover, the convergence of the block Gauss-Seidel method is independent of the grid refinement in the vicinity of a parabolic boundary layer. Here we prove this theoretically. Denote the error of the block iterative method after k iteration steps by

$$r_{ij}^{(k)} = u_{ij}^{(k)} - u_{ij}, \qquad i = 1, 2, \ldots, n-1.$$

We fix i and take the maximum of the modulus of $r_{ij}^{(k)}$ which is achieved at some j_0:

$$\left| r_{i,j_0}^{(k)} \right| = \max_{1\le j\le n-1} \left| r_{ij}^{(k)} \right|. \tag{3.19}$$

We subtract (3.13) from (3.18) and obtain

$$-d_{i,j_0}r_{i,j_0-1}^{(k+1)} + b_{i,j_0}r_{i,j_0}^{(k+1)} - e_{ij_0}r_{i,j_0+1}^{(k+1)} = a_{i,j_0}r_{i-1,j_0}^{(k+1)} + c_{i,j_0}r_{i+1,j_0}^{(k)}.$$

Rearranging some terms to the right-hand side and taking modulus of both sides, we have

$$b_{i,j_0}\left| r_{i,j_0}^{(k+1)} \right| \le d_{i,j_0}\left| r_{i,j_0-1}^{(k+1)} \right| + e_{i,j_0}\left| r_{i,j_0+1}^{(k+1)} \right| + a_{i,j_0}\left| r_{i-1,j_0}^{(k+1)} \right| + c_{i,j_0}\left| r_{i+1,j_0}^{(k)} \right|.$$

Using (3.19) we rewrite the last inequality in the form

$$(b_{i,j_0} - d_{i,j_0} - e_{i,j_0})\left| r_{i,j_0}^{(k+1)} \right| \le a_{i,j_0}\left| r_{i-1,j_0}^{(k+1)} \right| + c_{i,j_0}\left| r_{i+1,j_0}^{(k)} \right|. \tag{3.20}$$

Thus, we get

$$-a_{ij_0}\left| r_{i-1,j_0}^{(k+1)} \right| + s_i\left| r_{i,j_0}^{(k+1)} \right| - c_{ij_0}\left| r_{i+1,j_0}^{(k)} \right| \le 0 \tag{3.21}$$

where $s_i = b_{i,j_0} - d_{i,j_0} - e_{i,j_0}$. Let us introduce the notation

$$\rho_i = \frac{b_i}{\exp\sigma_i - 1}$$

where $\sigma_i = b_i h/\varepsilon$. Then $\dfrac{b_i \exp\sigma_i}{\exp\sigma_i - 1} = b_i + \rho_i$. With this notation, dividing the inequality (3.21) by $\dfrac{h_{j_0} + h_{j_0+1}}{2}$ we get

$$-(b_{i-1} + \rho_{i-1})\left|r^{(k+1)}_{i-1,j_0}\right| + (b_i + \rho_{i-1} + \rho_i)\left|r^{(k+1)}_{i,j_0}\right| - \rho_i\left|r^{(k)}_{i+1,j_0}\right| \le 0. \quad (3.22)$$

Then we consider the k-th iteration step of the majorized Gauss-Seidel process

$$-(b_{i-1} + \rho_{i-1})t^{(k+1)}_{i-1} + (b_i + \rho_{i-1} + \rho_i)t^{(k+1)}_i - \rho_i t^{(k)}_{i+1} = 0,\ i = 1, 2, \ldots, n-1,$$
$$t^{(k+1)}_0 = t^{(k+1)}_n = 0. \quad (3.23)$$

Lemma 30. *Let the inequality*

$$\left|r^{(k)}_{i,j_0}\right| \le t^{(k)}_i$$

be valid for all $i = 1, 2, \ldots, n-1$. Then the estimate

$$\left|r^{(k+1)}_{i,j_0}\right| \le t^{(k+1)}_i \quad \forall\, i = 1, 2, \ldots, n-1 \quad (3.24)$$

holds.

Proof. Because of (3.21) we have

$$\left|r^{(k+1)}_{i,j_0}\right| \le \frac{b_{i-1} + \rho_{i-1}}{b_i + \rho_{i-1} + \rho_i}\left|r^{(k+1)}_{i-1,j_0}\right| + \frac{\rho_i}{b_i + \rho_{i-1} + \rho_i}\left|r^{(k)}_{i+1,j_0}\right|, \quad i = 1, 2, \ldots, n-1.$$

Taking into account (3.23) we get

$$t^{(k+1)}_i = \frac{b_{i-1} + \rho_{i-1}}{b_i + \rho_{i-1} + \rho_i}t^{(k+1)}_{i-1} + \frac{\rho_i}{b_i + \rho_{i-1} + \rho_i}t^{(k)}_{i+1} \quad i = 1, 2, \ldots, n-1.$$

Now we use induction on i.

1. For $i = 1$ we have
$$t^{(k+1)}_1 = \frac{\rho_1}{b_1 + \rho_0 + \rho_1}t^{(k)}_2$$
and
$$\left|r^{(k+1)}_{1,j_0}\right| = \frac{\rho_1}{b_1 + \rho_0 + \rho_1}\left|r^{(k)}_{2,j_0}\right| \le \frac{\rho_1}{b_1 + \rho_0 + \rho_1}t^{(k)}_2 = t^{(k+1)}_1.$$

2. Let the statement (3.24) be valid for $i \le m-1$. Then we obtain

$$\begin{aligned}\left|r^{(k+1)}_{m,j_0}\right| &= \frac{b_{m-1}+\rho_{m-1}}{b_m+\rho_{m-1}+\rho_m}\left|r^{(k+1)}_{m-1,j_0}\right| + \frac{\rho_m}{b_m+\rho_{m-1}+\rho_m}\left|r^{(k)}_{m+1,j_0}\right| \\ &\le \frac{b_{m-1}+\rho_{m-1}}{b_m+\rho_{m-1}+\rho_m}t^{(k+1)}_{m-1} + \frac{\rho_m}{b_m+\rho_{m-1}+\rho_m}t^{(k)}_{m+1} = t^{(k+1)}_m .\end{aligned}$$

The proof of the lemma is completed. □

Thus, the convergence estimate of the block Gauss-Seidel method coincides with that of the pointwise Gauss-Seidel method for an ordinary differential equation and is independent of the grid refinement in the y-direction.

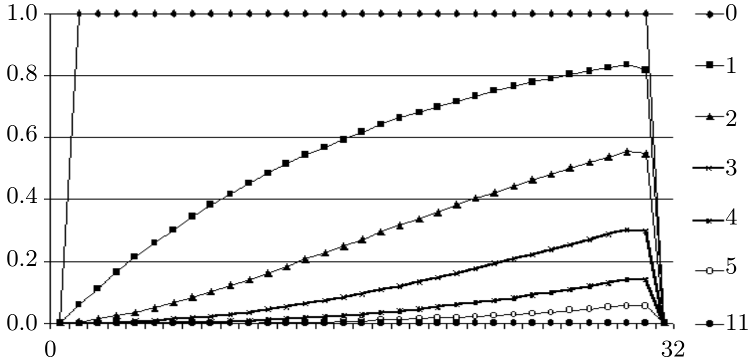

Fig. 9: The error of the pointwise Gauss-Seidel method $r^{(k)}_{i,0.5}$.

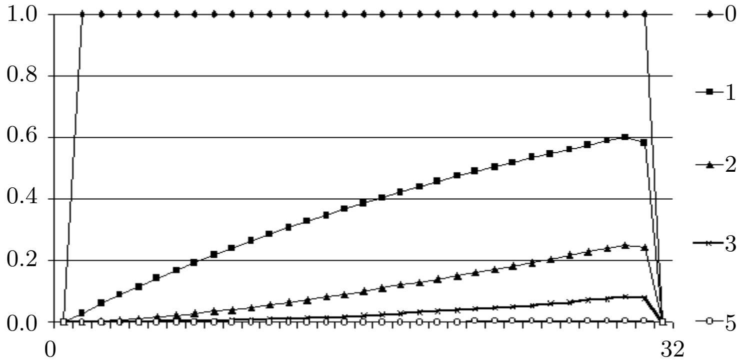

Fig. 10: The error of the block Gauss-Seidel method $r^{(k)}_{i,0.5}$.

First, we investigate numerically the convergence of the pointwise and block Gauss-Seidel methods for a model problem free of a boundary layer on a uniform grid.

We consider the problem

$$-\varepsilon \Delta u + \partial_1 u = 0 \quad \text{in} \quad \Omega, \qquad u = 1 \quad \text{on} \quad \Gamma.$$

It has the exact solution $u \equiv 1$.

In the Figures 9 and 10 the behaviour of the error

$$s_{i,0.5}^{(k)} = \left| u(x_i, 0.5) - u^{(k)}(x_i, 0.5) \right|$$

along the middle line $y = 0.5$ after k iteration steps is demonstrated for the pointwise and block Gauss-Seidel methods respectively. As the initial guess we take

$$u_{ij}^{(0)} = 0, \quad i, j = 1, ..., n-1 \qquad \text{and} \qquad r_{ij}^{(0)} = 1, \quad i, j = 1, ..., n-1.$$

The use of the cascadic algorithm allows to improve further the convergence. With this approach we take the interpolation of the solution on a coarse grid as the initial guess on the finer grid with the halved mesh size.

Now we consider the construction of the interpolation from a coarse grid to a finer one.

In the numerical experiments we applied the linear interpolation in the y-direction

$$u^I(x_i, y_k^*) = \frac{y_j - y_k^*}{h_j} u_{i,j-1} + \frac{y_k^* - y_{j-1}}{h_j} u_{i,j} \tag{3.25}$$

where $y_k^* \in [y_{j-1}, y_j]$, $h_j = y_j - y_{j-1}$. Let us show that with this interpolation the order of accuracy holds when the nodes of the grid are defined by (3.9).

To do this, we rewrite (3.25) in the form

$$u^I(x_i, y_k^*) = \alpha u_{i,j-1} + (1-\alpha) u_{i,j}, \quad \alpha = (y_j - y_k^*)/h_j.$$

Using the Taylor expansion with the second-order reminder term for $u_{i,j-1}$ and $u_{i,j}$ about (x_i, y_k^*), we have

$$\begin{aligned} \left| u^I(x_i, y_k^*) - u(x_i, y_k^*) \right| &\le \left| \alpha u_{i,j-1} + (1-\alpha) u_{i,j} - u(x_i, y_k^*) \right| \\ &\le \frac{1}{2} \left(\alpha (y_k^* - y_{j-1})^2 + (1-\alpha)(y_j - y_k^*)^2 \right) \left| \partial_{22} u(x_i, y_k^*) \right| \\ &\le c_1 h_j^2 \left| \partial_{22} u(x_i, y_k^*) \right| \le c_2 h^2. \end{aligned} \tag{3.26}$$

Here we used the estimate (2.131) from Lemma 24 and the definition (3.9) of the grid.

Note that the estimate (3.26) for the Shishkin grid has the form [57]

$$\left|u^I(x_i, y_k^*) - u(x_i, y_k^*)\right| \le c_3 h^2 \ln^2(1/h).$$

In the x-direction the grid is uniform. With decreasing the mesh size from $2h$ to h, we transform the grid equations (3.13) to determine the values of $u(x_i, y_j)$ for $i = 2m-1$, $m = 1, 2, ..., [n/2]$, $j = 1, 2, ..., n-1$:

$$-d_{ij}u^I_{i,j-1} + b_{ij}u^I_{ij} - c_{ij}u^I_{i,j+1} = f_{ij} + a_{ij}u_{i-1,j} + c_{ij}u_{i+1,j}.$$

The values of $u_{i-1,j}$ and $u_{i+1,j}$ at each level $i = 2k-1$ are known from the previous grid. Therefore to determine the values of $u(x_{2m-1}, y_j)$ at each level m one has to solve the system of linear algebraic equations with the tridiagonal matrix B_{2k-1}, $m = 1, 2, ..., [n/2]$. We show that in this case the order of convergence also holds.

Consider the error

$$\delta_{ij} = \left|u_{ij} - u^I_{ij}\right|, \quad i = 2m-1, \quad m = 1, 2, ..., [n/2], \quad j = 1, 2, ..., n-1.$$

It satisfies the system of equations

$$-d_{ij}\delta_{i,j-1} + b_{ij}\delta_{ij} - e_{ij}\delta_{i,j+1} = \theta_{ij}, \quad |\theta_{ij}| \le c_5 h^2$$

for $i = 2m-1$, $m = 1, 2, ..., [n/2]$, $j = 1, 2, ..., n-1$. Then we have

$$(a_{ij} + c_{ij}) \max_{i,j=1,2,\ldots,n-1} |\delta_{ij}| \le \max_{i,j=1,2,\ldots,n-1} |\theta_{ij}|.$$

Taking into account the definitions of a_{ij} and c_{ij} we get

$$\max_{i,j=1,2,\ldots,n-1} |\delta_{ij}| \le c_6 h^2.$$

The number of iteration step that is required to achieve the convergence criterion is shown in Table 7 for $\varepsilon = 1/2560$ on $n \times n$ grids. As the convergence criterion we used the following restriction on the residual after k iteration steps:

$$\max_{i,j=1,2,\ldots,n-1} \left| \left(L^h u^{(k)}\right)_{ij} - f_{ij} \right| \le \Delta^h.$$

We put

$$\Delta^h = 10^{-5} H^2 \cdot 2^{1-H/h}$$

where $1/H$ is the nodes of the coarser grid.

Table 7: The number of iteration step in the Gauss-Seidel method.

n	Gauss-Seidel method					
	pointwise		block		cascadic algorithm	
	Bakhvalov	Shishkin	Bakhvalov	Shishkin	Bakhvalov	Shishkin
32	28	15	2	2	2	2
64	97	31	2	2	2	2
128	355	70	3	3	2	2
256	1307	178	5	5	4	4
512	*	896	19	19	14	15
1024	*	*	151	152	124	125
2048	*	*	1052	1049	877	891
* – convergence was not achieved after 2500 iteration steps						

3.5 Discussion of the numerical results

We write the solution of the problem (3.3)–(3.4) as the series

$$u = \sum_{n=1}^{\infty} \gamma_n \psi_n(x) \sin(\pi n y) = S_{sol} \tag{3.27}$$

where

$$\psi_n(x) = C_{1n} \exp(\lambda_1^n x) + C_{2n} \exp(\lambda_2^n x) - 1, \tag{3.28}$$

$$C_{1n} = \frac{\exp(\lambda_2^n) - 1}{\exp(\lambda_2^n) - \exp(\lambda_1^n)}, \quad C_{2n} = \frac{1 - \exp(\lambda_1^n)}{\exp(\lambda_2^n) - \exp(\lambda_1^n)}, \tag{3.29}$$

$$\lambda_1^n = \frac{1 + \sqrt{1 + (2\varepsilon\pi n)^2}}{2\varepsilon}, \quad \lambda_2^n = \frac{1 - \sqrt{1 + (2\varepsilon\pi n)^2}}{2\varepsilon}, \tag{3.30}$$

$$\gamma_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ -\dfrac{4}{\varepsilon(\pi n)^3}, & \text{if } n \text{ is odd.} \end{cases} \tag{3.31}$$

Lemma 31. *The series* (3.27) *converges uniformly for* $x \in [0, 1]$.

Proof. Consider the sequence of the functions $\{\psi_n(x)\}_{n=1}^{\infty}$ and show that it is uniformly bounded on $x \in [0, 1]$.

Let us calculate the derivatives $\psi_n'(x)$, $\psi_n''(x)$:

$$\psi_n'(x) = \lambda_1^n C_{1n} \exp(\lambda_1^n x) + \lambda_2^n C_{2n} \exp(\lambda_2^n x),$$

$$\psi_n''(x) = \lambda_1^{n2} C_{1n} \exp(\lambda_1^n x) + \lambda_2^{n2} C_{2n} \exp(\lambda_2^n x).$$

Because of (3.29) and (3.30) the following inequalities hold:

$$\begin{array}{lll} \lambda_1^n \geq 0, & \lambda_2^n \leq 0 & \forall\, n = 1, 2, \dots\,, \\ C_{1n} \leq 0, & C_{2n} \leq 0 & \forall\, n = 1, 2, \dots\,. \end{array}$$

Since

$$\psi_n''(x) \geq 0 \qquad \forall\, x \in [0,1] \quad \forall\, n = 1, 2, \dots\,,$$

$\psi_n(x)$ is convex function on $[0,1]$. At the point of maximum the equality

$$\psi_n'(x_0) = \lambda_1^n C_{1n} \exp(\lambda_1^n x_0) + \lambda_2^n C_{2n} \exp(\lambda_2^n x_0) = 0$$

is valid. Then we have

$$\exp(x_0) = \left(\frac{\lambda_2^n}{\lambda_1^n} \frac{\exp(\lambda_1^n) - 1}{\exp(\lambda_2^n) - 1} \right)^{1/(\lambda_1^n - \lambda_2^n)}.$$

Calculate $\lim\limits_{n\to\infty} |\psi_n(x_0)|$. Let n be sufficiently large, for example, $2\varepsilon\pi n >> 1$, then $\lambda_1^n \approx \pi n$, and $\lambda_2^n \approx -\pi n$. It is easy to calculate that

$$\exp(x_0) = \left(-\frac{\exp(\pi n) - 1}{\exp(-\pi n) - 1} \right)^{1/2\pi n} = \exp\left(1/2\right).$$

Then we get

$$\begin{aligned} \psi_n(x_0) &= \frac{1}{\exp(\pi n) + 1} \exp(\pi n/2) + \frac{\exp(\pi n)}{\exp(\pi n) + 1} \exp(-\pi n/2) - 1 \\ &= \frac{2}{\exp(\pi n/2) + \exp(-\pi n/2)} - 1. \end{aligned}$$

This yields

$$\lim_{n\to\infty} |\psi_n(x_0)| = 1. \tag{3.32}$$

Thus, the sequence $\{\psi_n(x)\}_{n=1}^{\infty}$ is uniformly bounded on $[0,1]$, in other words, there exists such a constant M that

$$|\psi_n(x)| \leq M \qquad \forall\, x \in [0,1] \quad \forall\, n = 1, 2, \dots\,.$$

The sequence of the functions $\{\sin(\pi n y)\}_{n=1}^{\infty}$ is uniformly bounded on $[0,1]$ by 1.

Therefore, the terms of the series (3.27) satisfy the inequality

$$|\gamma_n \psi_n(x) \sin(\pi n y)| \leq M\gamma_n, \quad \forall\, n = 1, 2, \dots \tag{3.33}$$

where γ_n are the terms of the convergent series

$$S = -\frac{4M}{\varepsilon\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3} \tag{3.34}$$

of numbers. Then according to the Weierstrass criterion of the uniform convergence of functional series, the series (3.27) uniformly converges.□

The estimate (3.33) shows that the series (3.27) converges at least as (3.34). We denote by S_K the partial sum of (3.34). The following estimate holds (see [53]):

$$|S - S_K| \le \frac{4M}{\varepsilon\pi^3}\,\frac{1}{K^2}.$$

Therefore to achieve the given accuracy δ it is necessary to take at most K terms where

$$K = 2\sqrt{\frac{M}{\varepsilon\pi^3}\delta}.$$

From (3.32) we have that the constant M is close to 1.

In the numerical experiments the series was calculated within an accuracy $\delta = 10^{-5}$. The exact solution was calculated as the partial sums S_{sol}^{1000}, S_{sol}^{2000}, and S_{sol}^{3000}. In Tables 8, 9, and Figures 11, 12 the numerical results are presented on the sequence of grids for $\varepsilon = 10^{-3},\ 10^{-2}$. We use the notations

$$R_{abs}^{n,K} = \max_{i,j=0,\dots,n}\left|u_{ij}^n - S_{sol}^K(x_i,y_j)\right|.$$

Here u_{ij}^n is the solution of the discrete problem at the node (x_i, y_j) of the $(n+1)\times(n+1)$ grid, K is the number of the terms of the series. In the Figures 11, 12 the values of $R_{abs}^{n,3000}(n)$ are marked by the numbers 2, 3, and 4 for the Shishkin, the Bakhvalov grids and the grid (3.9) respectively. For comparison the straight lines with slapes $\mathrm{tg}\varphi = 1$ and $\mathrm{tg}\varphi = 2$ marked by 1 are shown in Figures 12 and 11 respectively. For $\varepsilon = 10^{-2}$ the method has the second-order convergence. When ε decreases to 10^{-3}, the method becomes first-order convergent.

Finally, we discuss the results obtained in the two-dimensional case with the special approximation of the right-hand side similar to that considered in Chapter 1. We considered the Dirichlet problem

$$\begin{aligned} -\varepsilon\Delta u + \partial_1((1+2x)u) = f \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{on} \quad \Gamma \end{aligned}$$

where

$$f = 6x^2 + 2x - 2\varepsilon + 2d, \quad d = \frac{\exp(-2/\varepsilon)}{1-\exp(-2/\varepsilon)}.$$

Table 8: The error $R^{n,K}_{abs}$ for $\varepsilon = 10^{-3}$.

n	Bakhvalov grids			Shishkin grids		
	$R^{n,1000}_{abs}$	$R^{n,2000}_{abs}$	$R^{n,3000}_{abs}$	$R^{n,1000}_{abs}$	$R^{n,2000}_{abs}$	$R^{n,3000}_{abs}$
32	7.160_{10}-3	7.160_{10}-3	7.160_{10}-3	7.224_{10}-3	7.223_{10}-3	7.223_{10}-3
64	1.545_{10}-3	1.546_{10}-3	1.546_{10}-3	2.993_{10}-3	2.992_{10}-3	2.992_{10}-3
128	6.824_{10}-4	6.812_{10}-4	6.812_{10}-4	1.165_{10}-3	1.164_{10}-3	1.164_{10}-3
256	2.471_{10}-4	2.460_{10}-4	2.459_{10}-4	4.013_{10}-4	4.008_{10}-4	4.009_{10}-4
512	7.389_{10}-5	7.200_{10}-5	7.188_{10}-5	1.209_{10}-4	1.203_{10}-4	1.202_{10}-4
1024	2.819_{10}-5	1.909_{10}-5	1.889_{10}-5	3.634_{10}-5	3.379_{10}-5	3.374_{10}-5

Table 9: The error $R^{n,K}_{abs}$ for $\varepsilon = 10^{-2}$.

n	Bakhvalov grids			Shishkin grids		
	$R^{n,1000}_{abs}$	$R^{n,2000}_{abs}$	$R^{n,3000}_{abs}$	$R^{n,1000}_{abs}$	$R^{n,2000}_{abs}$	$R^{n,3000}_{abs}$
32	1.755_{10}-3	1.755_{10}-3	1.755_{10}-3	3.629_{10}-3	3.629_{10}-3	3.629_{10}-3
64	4.896_{10}-4	4.896_{10}-4	4.896_{10}-4	9.579_{10}-4	9.579_{10}-4	9.579_{10}-4
128	1.254_{10}-4	1.254_{10}-4	1.254_{10}-4	2.435_{10}-4	2.435_{10}-4	2.435_{10}-4
256	3.158_{10}-5	3.159_{10}-5	3.159_{10}-5	6.113_{10}-5	6.111_{10}-5	6.111_{10}-5
512	7.916_{10}-6	7.919_{10}-6	7.920_{10}-6	1.531_{10}-5	1.530_{10}-5	1.530_{10}-5
1024	2.614_{10}-6	2.120_{10}-6	2.120_{10}-6	3.987_{10}-6	3.977_{10}-6	3.977_{10}-6

Table 10: The error $R^{n,K}_{abs}$ on the grid (3.9).

n	$\varepsilon = 10^{-2}$			$\varepsilon = 10^{-3}$		
	$R^{n,1000}_{abs}$	$R^{n,2000}_{abs}$	$R^{n,3000}_{abs}$	$R^{n,1000}_{abs}$	$R^{n,2000}_{abs}$	$R^{n,3000}_{abs}$
32	1.54_{10}-3	1.54_{10}-3	1.54_{10}-3	3.22_{10}-3	3.22_{10}-3	3.22_{10}-3
64	4.46_{10}-4	4.46_{10}-4	4.46_{10}-4	1.57_{10}-3	1.57_{10}-3	1.57_{10}-3
128	1.16_{10}-4	1.16_{10}-4	1.16_{10}-4	6.85_{10}-4	6.85_{10}-4	6.85_{10}-4
256	2.95_{10}-5	2.95_{10}-5	2.95_{10}-5	2.48_{10}-4	2.46_{10}-4	2.46_{10}-4
512	7.43_{10}-6	7.39_{10}-6	7.39_{10}-6	7.37_{10}-5	7.20_{10}-5	7.19_{10}-5
1024	2.55_{10}-6	1.85_{10}-6	1.85_{10}-6	2.83_{10}-5	1.90_{10}-5	1.89_{10}-5

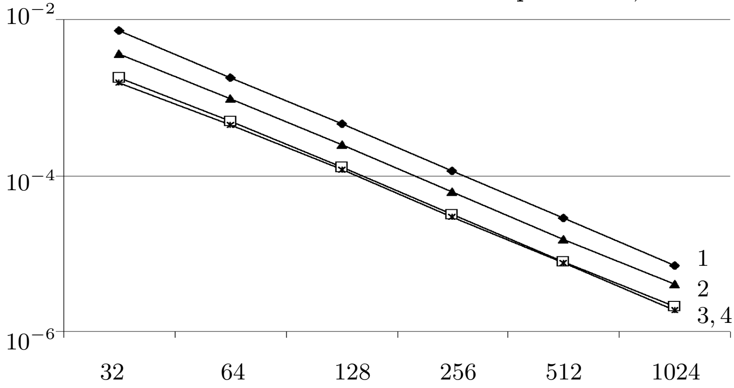

Fig. 11: The error $R_{abs}^{n,3000}(n)$ for $\varepsilon = 10^{-2}$.

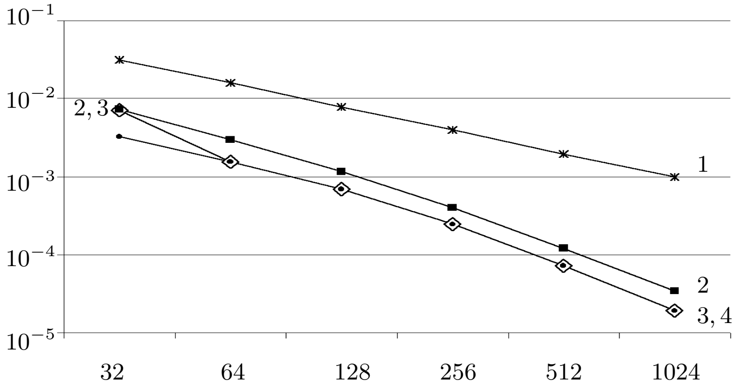

Fig. 12: The error $R_{abs}^{n,3000}(n)$ for $\varepsilon = 10^{-3}$.

The solution of this problem has the parabolic boundary layer near the boundary Γ_{tg} and the regular one near Γ_{out}.

Table 11: The error R_{abs}^{n} for standard and special approximations of the right-hand side.

approximation of the right-hand side	n					
	32	64	128	256	512	1024
standard	3.99_{10}-2	1.99_{10}-2	9.68_{10}-3	4.52_{10}-3	1.93_{10}-3	6.27_{10}-4
special	4.32_{10}-3	2.65_{10}-3	1.36_{10}-3	6.26_{10}-4	2.33_{10}-4	1.40_{10}-4

Table 11 contains the results obtained on the Bakhvalov grid with the fitted quadrature rule with the special and standard approximations of the right-hand side for $\varepsilon = 1/2560$. The results demonstrate that the application of the special quadrature rule for the approximation of the right-hand side improves the accuracy.

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