A two-dimensional nonuniform difference scheme with higher order of accuracy

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Introduction

The present paper is devoted to construction and justification of *nonuniform* difference schemes of higher orders of accuracy for two-dimensional boundary-value problem for elliptic type equation on a rectangle. The general idea of construction of such scheme is similar to that in the paper [1], where it is stated for ordinary differential equation, but the increase of dimensionality has complicated both the scheme and the proof of its accuracy. Nevertheless, the fourth order of accuracy in uniform norm is proved for the constructed scheme, and this fact is illustrated with numerical examples.

As it is in one-dimensional case, the difference scheme is similar in structure to the system of the method of extrapolated equations by U. Rüde [2] for finite elements. However, the proof of accuracy of the constructed scheme differs from substantiation of U. Rüde method based on minimization of functional.

Let recall that the standard difference method with the second order of accuracy on a rectangle gives a system of linear algebraic equations with five-diagonal matrix under corresponding ordering of unknowns. The scheme constructed here results in a system of equations with nine-diagonal matrix preserving the basic properties: positive definiteness, symmetry and positive invertibility.

Let also recall that the term “nonuniform scheme” had appeared due to different rules of construction of grid equations in neighbouring nodes as distinct from uniform schemes [3], where the rule of construction is the same for all nodes of the grid.

1 Boundary-value problem and its nonuniform difference approximation

Let Ω be unit square $(0,1)\times(0,1)$ with boundary Γ. Consider a boundary-value problem

$$-\Delta u + du = f \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = g \quad \text{on} \quad \Gamma \tag{1.2}$$

with smooth enough given functions

$$d, f \in C^4(\overline{\Omega}), \tag{1.3}$$

$$d \geq 0 \quad \text{in} \quad \overline{\Omega}. \tag{1.4}$$

These conditions ensure unique solvability of the problem. Suppose the solution to be smooth enough:

$$u \in C^6(\overline{\Omega}). \tag{1.5}$$

For difference approximation of the problem (1.1) — (1.2) construct an uniform difference grid

$$\overline{\omega}_h = \{z_{i,j} = (x_i, y_j): \; x_i = ih, \; y_j = jh, \; i = 0, 1, \ldots, n, \; j = 0, 1, \ldots, n\}$$

with the step $h = 1/n$ and *even* $n \geq 4$. Also, introduce the set of inner nodes

$$\omega_h = \{z_{i,j} \in \overline{\omega}_h \; : \; i = 1, 2, \ldots, n-1, \; j = 1, 2, \ldots, n-1\}$$

and divide it into the sets of nodes only with even indices, only with odd indices and with indices of different evenness (the first index is even and the second is odd, or vice versa):

$$\begin{aligned}
&\overline{\omega}_{00} = \{z_{i,j} \in \overline{\omega}_h: \; i = 0, 2, \ldots, n, \; j = 0, 2, \ldots, n\}, \; \omega_{00} = \overline{\omega}_{00} \setminus \Gamma, \\
&\omega_{11} = \{z_{i,j} \in \omega_h: \; i = 1, 3, \ldots, n-1, \; j = 1, 3, \ldots, n-1\}, \\
&\overline{\omega}_{01} = \{z_{i,j} \in \overline{\omega}_h: \; i = 0, 2, \ldots, n, \; j = 1, 3, \ldots, n-1\}, \; \omega_{01} = \overline{\omega}_{01} \setminus \Gamma, \\
&\overline{\omega}_{10} = \overline{\omega}_h \setminus (\overline{\omega}_{00} \cup \omega_{11} \cup \overline{\omega}_{01}), \; \omega_{10} = \overline{\omega}_{10} \setminus \Gamma.
\end{aligned}$$

The standard finite difference approximation of the equation (1.1) consists in change of the second derivatives with respect to x and y with the second central differences

$$\begin{aligned}
u_{\overset{\circ}{x}\overset{\circ}{x}}(x,y) &= (u(x-h,y) - 2u(x,y) + u(x+h,y))/h^2, \\
u_{\overset{\circ}{y}\overset{\circ}{y}}(x,y) &= (u(x,y-h) - 2u(x,y) + u(x,y+h))/h^2.
\end{aligned} \tag{1.6}$$

As a result, the following grid problem is obtained:

$$\begin{aligned} &L^h u^h = f \quad \text{in} \quad \omega_h, \\ &u^h = g \quad \text{on} \quad \gamma_h = \Gamma \cap \overline{\omega}_h, \end{aligned} \tag{1.7}$$

with the difference operator

$$L^h v(z) = -v_{\overset{\circ}{x}\overset{\circ}{x}}(z) - v_{\overset{\circ}{y}\overset{\circ}{y}}(z) + d(z)v(z). \tag{1.8}$$

The second order of approximation is established by Taylor-series expansion of the solution u [3], and on the basis of difference maximum principle [3] the stability of solution in the grid norm

$$\|v\|_{\infty,\bar{\omega}_h} = \max_{z \in \overline{\omega}_h} |v(z)|$$

is proved. On the whole, this gives convergence of the approximate solution u^h of the problem (1.7) to the exact solution u of the problem (1.1) – (1.2) with the second order of accuracy:

$$\|u^h - u\|_{\infty,\overline{\omega}_h} \leq c_1 h^2 \|u\|^{(4)}_{\infty,\overline{\Omega}} \; {}^{*)} \tag{1.9}$$

here the following denotation is used:

$$\|u\|^{(k)}_{\infty,\overline{\Omega}} = \sum_{0 \leq i+j \leq k} \left\| \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \right\|_{\infty,\overline{\Omega}}$$

with integer $k \geq 0$ and

$$\|u\|_{\infty,\overline{\Omega}} = \sup_{\overline{\Omega}} |u|.$$

For construction of a scheme of the fourth order introduce an operator with doubled step

$$\begin{aligned} L^{2h} v(x,y) \; &= -(v(x-2h,y) + v(x,y-2h) - 4v(x,y) \\ &+ v(x+2h,y) + v(x,y+2h))/4h^2 + d(x,y)v(x,y) \end{aligned}$$

only in even nodes ω_{00}.

With the preceding notations consider the difference problem

$$L^h u^h = f \quad \text{in} \quad \omega_h \setminus \omega_{00}, \tag{1.10}$$

$$L^h u^h - L^{2h} u^h = 0 \quad \text{in} \quad \omega_{00}, \tag{1.11}$$

$$u^h = g \quad \text{on} \quad \gamma_h. \tag{1.12}$$

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*) Here and below we denote by a symbol c_i with integer indices i various constants independent of x and h.

This grid problem as well as (1.7) contains $(n+1)^2$ unknowns and $(n+1)^2$ equations. In even nodes nine-point stencil is obtained (Fig. 1. b), and in other nodes the scheme has a standard five-point stencil (Fig. 1. a).

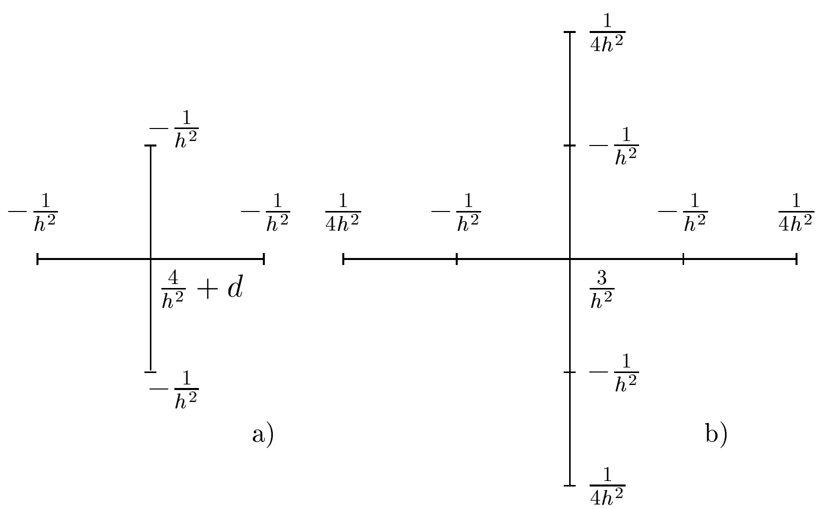

Fig. 1: Stencils of nonuniform difference scheme in even (b) and other (a) nodes.

For the functions defined on $\overline{\Omega}$ apply the denotation

$$v_{i,j} = v(x_i, y_j) = v(ih, jh).$$

In the equations (1.10) — (1.11) eliminate the boundary values (1.12). The remaining unknowns and equations number from 1 to $(n-1)^2$ in lexicographical order determined by the inner nodes $z_{1,1}$, $z_{1,2}, \ldots, z_{1,n-1}$, $z_{2,1}, \ldots, z_{n-1,n-1}$. As a result we obtain a system of linear algebraic equations with symmetric sparse matrix A^h

$$A^h U^h = F^h. \tag{1.13}$$

By way of illustration in Fig. 2 the structure of nonzero elements of the matrix A^h for the step h=1/8 is given.

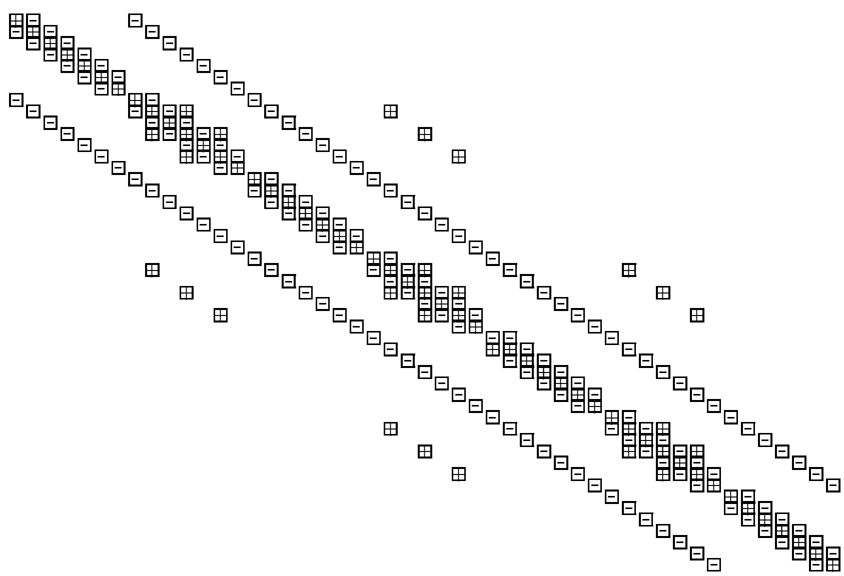

Fig. 2: Structure of nonzero elements of the matrix $A^{1/8}$.
The sign $\boxplus$ marks a positive element,
the sign $\boxminus$ marks negative one, and their absence implies zero element.

For theoretical consideration it is useful to write down the system (1.10)—(1.12) in vector form as well. To do this, number unknowns and equations from 1 to $(n+1)^2$ in lexicographical order determined by the nodes $z_{00},\ z_{01},\ldots,z_{0n},z_{10},\ldots,z_{nn}$. As a result, we obtain a system of linear algebraic equations with a matrix B^h

$$B^h V^h = G^h. \tag{1.14}$$

2 Stability and solvability of the grid problem

Let proof that matrix of the system (1.13) is positive definite.

Theorem 32. *If the condition* (1.4) *is satisfied, then the matrix A^h of the system* (1.13) *is positive definite.*

Proof. Multiply left part of each equation (1.10) and (1.11) by $hu^h(z)$ with corresponding z and sum over all $z \in \omega_h$:

$$h \sum_{z\in\omega_h} u^h(z) L^h u^h(z) - h \sum_{z\in\omega_{00}} u^h(z) L^{2h} u^h(z). \tag{2.1}$$

Set $u^h = 0$ on γ_h and for the obtained expression apply difference analog of the first Green function [3], going over to index notations:

$$\begin{aligned} h\sum_{z\in\omega_h} u^h(z)L^h u^h(z) &= h\sum_{i,j=1}^{n-1} d_{ij}(u^h_{ij})^2 \\ &+\frac{1}{h}\sum_{i,j=1}^{n}\left[(u^h_{ij}-u^h_{i-1,j})^2+(u^h_{ij}-u^h_{i,j-1})^2\right], \end{aligned} \tag{2.2}$$

$$\begin{aligned} 2h\sum_{z\in\omega_{00}} u^h(z)L^{2h} u^h(z) &= 2h\sum_{i,j=1}^{n/2-1} d_{2i,2j}(u^h_{2i,2j})^2 \\ &+\frac{1}{2h}\sum_{i,j=1}^{n/2}\left[(u^h_{2i,2j}-u^h_{2i-2,2j})^2+(u^h_{2i,2j}-u^h_{2i,2j-2})^2\right]. \end{aligned} \tag{2.3}$$

For real numbers a, b the equality $a^2+b^2 \geq (a+b)^2/2$ is true, from which follows that

$$\begin{aligned} &(u^h_{2i,2j}-u^h_{2i-1,2j})^2+(u^h_{2i,2j}-u^h_{2i,2j-1})^2 \\ &+(u^h_{2i-1,2j}-u^h_{2i-2,2j})^2+(u^h_{2i,2j-1}-u^h_{2i,2j-2})^2 \\ &\leq \frac{1}{2}\left[(u^h_{2i,2j}-u^h_{2i-2,2j})^2+(u^h_{2i,2j}-u^h_{2i,2j-2})^2\right]. \end{aligned} \tag{2.4}$$

With account of this inequality the expression (2.1) is estimated from below by the value

$$\begin{aligned} &\frac{3}{4h}\sum_{i,j=1}^{n}\left[(u^h_{i,j}-u^h_{i-1,j})^2+(u^h_{i,j}-u^h_{i,j-1})^2\right] \\ &+h\sum_{i,j=1}^{n-1} d_{i,j}(u^h_{i,j})^2 - h\sum_{i,j=1}^{n/2} d_{2i,2j}(u^h_{2i,2j})^2. \end{aligned} \tag{2.5}$$

The sum $h\sum_{i,j=1}^{n/2} d_{i,j}(u^h_{i,j})^2$ contains all the terms $h\sum_{i,j=1}^{n/2} d_{2i,2j}(u^h_{2i,2j})^2$. Therefore the difference

$$h\sum_{i,j=1}^{n/2} d_{i,j}(u^h_{i,j})^2 - h\sum_{i,j=1}^{n/2} d_{2i,2j}(u^h_{2i,2j})^2$$

is nonnegative. The first sum in (2.5) is estimated from below by means of the equation [3]

$$16h^2 \sum_{i,j=1}^{n} (u_{i,j}^h)^2 \le \sum_{i,j=1}^{n} \left[(u_{i,j}^h - u_{i-1,j}^h)^2 + (u_{i,j}^h - u_{i,j-1}^h)^2 \right], \tag{2.6}$$

which is an analog of embedding of norms from $H_0^1(\Omega)$ into $L^2(\Omega)$. Finally, the expression (2.5) is estimated from below by the value

$$12h \sum_{i,j=1}^{n-1} (u_{ij}^h)^2 = 12h \sum_{z \in \omega_h} (u^h(z))^2. \tag{2.7}$$

Comparing it with (2.1) we arrive at the statement of Theorem. □

Symmetry and positive definiteness of the matrix A^h lead to two useful conclusions. First, the system (1.13) has unique solution u^h for any right part F^h, which follows from inadmissibility of zero eigenvalue of the matrix A^h. Second, for approximate solution of the system (1.13) an application of a number of various direct and iterative methods [4] becomes possible.

Now, let show that the system (1.14) satisfies comparison theorems despite that it is not M-matrix. For this purpose introduce a denotation $G^h \le 0$ for the vector G^h with components G_j^h, $j = 1, \ldots, (n+1)^2$, which signifies component-wise comparison.

Theorem 33. *Let the condition* (1.4) *be satisfied and step* h *be small enough:*

$$h \le 2/(5\|d\|_{\infty,\overline{\Omega}}). \tag{2.8}$$

Then for the system (1.14) *from* $G^h \ge 0$ *the inequality* $V^h \ge 0$ *follows.*

Proof. In order to use standard results on M-matrices it is necessary that diagonal elements would be positive and off-diagonal ones do nonnegative. This condition is satisfied for equations in the nodes ω_{11}, ω_{10} and ω_{01}, but not for equations in the nodes ω_{00} (see Fig. 2). Therefore slightly transform the system (1.14) or, what is the same, the system (1.10) — (1.12) so that to get rid of positive off-diagonal elements in the nodes ω_{00}. For that, to each equation corresponding to $(x, y) \in \omega_{00}$ add four equations corresponding to the nodes $(x \pm h, y \pm h) \in \omega_{11}$ with a weight a and four equations corresponding to the nodes $(x \pm h, y) \in \omega_{10}, (x, y \pm h) \in \omega_{01}$ with a weight b. As a result, in a node $(x, y) \in \omega_{00}$ we obtain an equation with the stencil

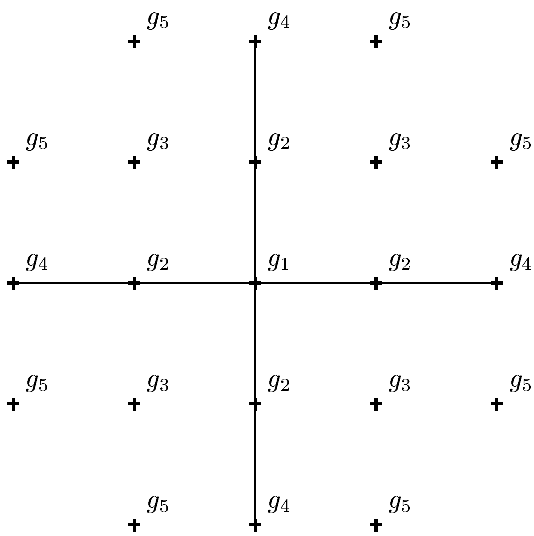

Fig. 3: 21-point stencil of the equation in a node $(x, y) \in \omega_{00}$ after transformation.

shown in Fig. 3., where

$$
\begin{aligned}
g_1 &= \frac{3}{h^2} - \frac{4b}{h^2},\\
g_2 &= -\frac{1}{h^2} + b\left(\frac{4}{h^2} + d\right) - \frac{2a}{h^2},\\
g_3 &= a\left(\frac{4}{h^2} + d\right) - \frac{2b}{h^2},\\
g_4 &= \frac{1}{4h^2} - \frac{b}{h^2},\\
g_5 &= -\frac{a}{h^2}.
\end{aligned}
\tag{2.9}
$$

Let try to choose the weights a, b so that in the equation obtained after transformation the diagonal element would be positive and off-diagonal elements do nonnegative. This will be true if

$$
g_1 \geq 0, \quad g_2 \leq 0, \quad g_3 \leq 0, \quad g_4 \leq 0, \quad g_5 \leq 0. \tag{2.10}
$$

This results in the problem to determinate the admissible state. Let for a step h the condition (2.8) be satisfied. Then the problem (2.10) has a nonempty set of admissible values, from which we choose

$$a = 1/20, \; b = 1/4. \tag{2.11}$$

Finally, the following coefficients of the stencil in Fig. 3 are obtained:

$$g_1 = \frac{2}{h^2}, \quad g_2 = -\frac{1}{10h^2} + \frac{d}{4},$$
$$g_3 = -\frac{3}{10h^2} + \frac{d}{20}, \quad g_4 = 0, \quad g_5 = -\frac{1}{20h^2}.$$

It is easy to verify that under the condition (2.8) we arrive at the inequalities (2.10). Thus, instead of (1.14) we obtain a system

$$\overline{B}^h V^h = \overline{G}^h \tag{2.12}$$

with M-matrix $\overline{B}^h$ and the same solution V^h. Due to positiveness of the weights $a, \; b$ the inequality $\overline{G}^h \geq 0$ is true. Therefore on the basis of the properties of M-matrices [3]

$$V^h \geq 0. \quad \square$$

Prove an a priori estimate useful for further reasonings.

Theorem 34. *Let for the problem*

$$\begin{aligned} &L^h v^h = g^h \quad in \quad \omega_h \setminus \omega_{00}, \\ &L^h v^h - L^{2h} v^h = g^h \quad in \quad \omega_{00}, \\ &v^h = g^h \quad on \quad \gamma_h \end{aligned} \tag{2.13}$$

the estimates (1.4) *and* (2.8) *be fulfilled. Then*

$$\|v^h\|_{\infty,\overline{\omega}_h} \leq \frac{11}{48} \|g^h\|_{\infty,\omega_h} + \|g^h\|_{\infty,\gamma_h}. \tag{2.14}$$

Proof. Introduce a function

$$w = c_3 + c_4 x(1-x) \quad \text{in} \quad \overline{\Omega} \tag{2.15}$$

with the constants

$$c_3 = \|g^h\|_{\infty,\gamma_h}, \quad c_4 = \frac{11}{12} \|g^h\|_{\infty,\omega_h}. \tag{2.16}$$

Note that

$$L^h w = Lw = dw + 2c_4 \geq 2c_4 \quad \text{in } \omega_h, \tag{2.17}$$

$$L^{2h} w = Lw = dw + 2c_4 \quad \text{in } \omega_{00}. \tag{2.18}$$

Therefore for the nodes $(x, y) \in \omega_h \setminus \omega_{00}$ we have

$$L^h w \geq 2c_4 \geq \left\| g^h \right\|_{\infty, \omega_h} \geq \left| g^h \right|. \tag{2.19}$$

For the boundary nodes $(x, y) \in \gamma_h$ it is also evident that

$$w \geq \left\| g^h \right\|_{\infty, \gamma_h} \geq \left| g^h \right|. \tag{2.20}$$

Consider grid operator in a node $(x, y) \in \omega_{00}$, which is transformed according to the rule pointed out in Theorem 2:

$$\begin{aligned}
&L^h w - L^{2h} w + a \left(L^h w(x+h, y+h) + L^h w(x-h, y+h) \right. \\
&\left. + L^h w(x+h, y-h) + L^h w(x-h, y-h) \right) + b \left(L^h w(x, y+h) \right. \\
&\left. + L^h w(x, y-h) + L^h w(x+h, y) + L^h w(x-h, y) \right) \\
&\geq 8ac_4 + 8bc_4 = \frac{12}{5} c_4 \geq \frac{11}{5} \left\| g^h \right\|_{\infty, \omega_h} \\
&\geq \left| g^h + a \left(g^h(x+h, y+h) + g^h(x-h, y+h) \right. \right. \\
&\left. + g^h(x+h, y-h) + g^h(x-h, y-h) \right) + b \left(g^h(x, y+h) \right. \\
&\left. \left. + g^h(x, y-h) + g^h(x+h, y) + g^h(x-h, y) \right) \right|.
\end{aligned} \tag{2.21}$$

Introduce vectors V^h and W^h with the components

$$V^h = \left\{ v_{ij}^h \right\}_{i,j=0}^{n+1}, \quad W^h = \left\{ w_{ij} \right\}_{i,j=0}^{n+1},$$

which are ordered as in the system (1.14). Then from the inequalities (2.19) — (2.21) it follows that

$$\overline{B}^h W^h \geq \overline{B}^h V^h, \quad \text{i.e.} \quad \overline{B}^h \left(W^h - V^h \right) \geq 0.$$

From the properties of M-matrices it follows that

$$W^h - V^h \geq 0, \quad \text{i.e.} \quad w \geq v^h \quad \text{in} \quad \overline{\omega}_h.$$

Similarly, from (2.19) — (2.21) it follows that

$$w \geq -v^h \quad \text{in} \quad \overline{\omega}_h.$$

Therefore

$$\left|v^h\right| \le w \quad \text{in} \quad \overline{\omega}_h. \tag{2.22}$$

In the left-hand side take maximum over $\overline{\omega}_h$, and in the right-hand side do over $\overline{\Omega}$. Finally we obtain

$$\left\|v^h\right\|_{\infty,\overline{\omega}_h} \le c_3 + c_4/4,$$

that is equivalent to (2.14). □

3 Convergence of the nonuniform difference scheme

Theorem 35. *Let* u, u^h *be solutions of the problems* (1.1)–(1.2) *and* (1.10)–(1.12)*, respectively, and the conditions* (1.3) — (1.5) *be satisfied. Then*

$$\|u - u^h\|_{\infty,\overline{\omega}_h} \le c_5 h^4. \tag{3.1}$$

Proof. We will establish a finer structure of the error. Let prove that the solution u^h can be represented as

$$u^h = u + h^4\rho^h \qquad \text{in} \quad \omega_{11}, \tag{3.2}$$

$$u^h = u + w_{01}h^4 + h^4\rho^h \quad \text{in} \quad \overline{\omega}_{01} \cup \overline{\omega}_{10}, \tag{3.3}$$

$$u^h = u + w_{00}h^4 + h^4\rho^h \qquad \text{in} \quad \overline{\omega}_{00}, \tag{3.4}$$

where the functions

$$w_{01} = -\frac{1}{48}\mu, \quad w_{00} = -\frac{1}{12}\mu, \quad \mu = \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \tag{3.5}$$

does not depend on h, and the remainder term ρ^h is limited in the following way:

$$\left\|\rho^h\right\|_{\infty,\overline{\omega}_h} \le c_6. \tag{3.6}$$

In the expression (1.7), apply Taylor series expansion from the points $(x \pm h, y)$ and $(x, y \pm h)$ into the node (x, y). Further we omit the argument (x, y) if this does not arouse misunderstanding:

$$\begin{aligned} u_{\overset{\circ}{x}\overset{\circ}{x}} &= \frac{\partial^2 u}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} + h^4\mu^h_{1x}, \\ u_{\overset{\circ}{y}\overset{\circ}{y}} &= \frac{\partial^2 u}{\partial y^2} + \frac{h^2}{12}\frac{\partial^4 u}{\partial y^4} + h^4\mu^h_{1y}, \end{aligned} \tag{3.7}$$

where

$$|\mu_{1x}^h| \leq \frac{1}{360}\left\|\frac{\partial^6 u}{\partial x^6}\right\|_{\infty,\overline{\Omega}} \quad \text{in} \quad \omega_h,$$
$$|\mu_{1y}^h| \leq \frac{1}{360}\left\|\frac{\partial^6 u}{\partial y^6}\right\|_{\infty,\overline{\Omega}} \quad \text{in} \quad \omega_h. \tag{3.8}$$

With consideration of the expansions (3.2), (3.4) and (3.7) for odd nodes ω_{11} we obtain

$$\begin{aligned} L^h u^h = \; & L^h u + h^4 L^h \rho^h - h^2(w_{01}(x+h,y) \\ & + w_{01}(x-h,y) + w_{01}(x,y+h) + w_{01}(x,y-h)). \end{aligned} \tag{3.9}$$

For the function w_{01} use Taylor series expansion from the points $(x \pm h, y)$ and $(x, y \pm h)$ into the node (x,y):

$$\begin{aligned} w_{01}(x+h,y) + w_{01}(x-h,y) + w_{01}(x,y+h) + w_{01}(x,y-h) \\ = 4w_{01} + 2h^2 \mu_{01}^h, \end{aligned} \tag{3.10}$$

where with account of (3.5) we have

$$2\left|\mu_{01}^h\right| \leq \left\|\frac{\partial^2 \omega_{01}}{\partial x^2} + \frac{\partial^2 \omega_{01}}{\partial y^2}\right\|_{\infty,\overline{\Omega}} = \frac{1}{124}\|u\|_{\infty,\overline{\Omega}}^{(6)}. \tag{3.11}$$

Taking into consideration the expansions (3.7), (3.10) into (3.9), we obtain the equality

$$\begin{aligned} & L^h u^h = (-\Delta u + du) - \frac{h^2}{12}\mu \\ & -h^4(\mu_{1x}^h + \mu_{1y}^h) + h^4 L^h \rho^h - 4h^2 \omega_{01} - 2h^4 \mu_{01}. \end{aligned}$$

On the basis of equations (1.1), (1.10) and definitions (3.5) a cancellation of terms of the orders 1 and h^2 is performed. Divide the remaining terms by h^4. As a result we arrive at the inequality

$$L^h \rho^h = \mu_{1x}^h + \mu_{1y}^h + 2\mu_{01}^h \quad \text{in} \quad \omega_{11}. \tag{3.12}$$

Substitution of the expansions (3.3), (3.4), (3.10) into the grid operator (1.11) for even nodes ω_{00} gives the following:

$$\begin{aligned} & L^h u^h - L^{2h} u^h = L^h u - L^{2h} u + h^4(L^h \rho^h - L^{2h} \rho^h) \\ & + h^4\left(4w_{00}/h^2 + dw_{00}\right) - 4h^2 w_{01} - 2h^4 \mu_{01}^h. \end{aligned} \tag{3.13}$$

In odd nodes w_{00} expressions similar to (3.7) are valid, but with doubled step, which gives

$$L^{2h}u = (-\Delta u + du) - \frac{h^2}{3}\mu - h^4(\mu^h_{2x} + \mu^h_{2y}), \tag{3.14}$$

where

$$\left|\mu^h_{2x}\right| \le \frac{4}{45}\left\|\frac{\partial^6 u}{\partial x^6}\right\|_{\infty,\overline{\Omega}}, \quad \left|\mu^h_{2y}\right| \le \frac{4}{45}\left\|\frac{\partial^6 u}{\partial y^6}\right\|_{\infty,\overline{\Omega}} \quad \text{in} \quad \omega_{00}. \tag{3.15}$$

Taking account of (3.14) in (3.13), we obtain the equality

$$\begin{aligned} L^h u^h - L^{2h} u^h \ &= h^4(L^h\rho^h - L^{2h}\rho^h) + 4h^2 w_{00} + dw_{00}h^4 - 4h^2 w_{01} - 2h^4\mu^h_{01} \\ &- \frac{h^2}{12}\mu - h^4\mu^h_{1x} - h^4\mu^h_{1y} + \frac{h^2}{3}\mu + h^4\mu^h_{2x} + h^4\mu^h_{2y}. \end{aligned}$$

Again, on the basis of equations (1.1), (1.10) and definitions (3.5) the cancellation of terms of the orders 1 and h^2 takes place. In this case the cancellation of the terms of order h^2 is performed due to proper choice of multiplier at $L^{2h}u^h$. The remaining terms after division by h^4 give the equality

$$L^h\rho^h - L^{2h}\rho^h = \mu^h_{1x} + \mu^h_{1y} - \mu^h_{2x} - \mu^h_{2y} - dw_{00} \quad \text{in} \quad \omega_{00}. \tag{3.16}$$

Substitution of the expansions (3.2), (3.3), (3.4) into the grid operator (1.10) for nodes with alternating evenness of indices w_{10} gives the relation

$$\begin{aligned} &L^h u^h = L^h u + h^4 L^h \rho^h \\ &+h^4\left(\left(\frac{4w_{01}}{h^2} + dw_{01}\right) - \frac{w_{00}(x+h,y)}{h^2} - \frac{w_{00}(x-h,y)}{h^2}\right). \end{aligned} \tag{3.17}$$

For the function w_{00} apply Taylor series expansion from the points $(x \pm h, y)$ into the node (x, y) similarly to (3.10), (3.11). This yields the equality:

$$w_{00}(x+h,y) + w_{00}(x-h,y) = 2w_{00}(x,y) + h^2\mu^h_{00}(x,y), \tag{3.18}$$

where with account of (3.5) we have

$$\left|\mu^h_{00}\right| \le \frac{1}{6}\|u\|^{(6)}_{\infty,\overline{\Omega}}. \tag{3.19}$$

Taking into account the expansions (3.7), (3.18) into (3.17), we obtain the equality

$$\begin{aligned} L^h u^h \ &= (-\Delta u + du) - \frac{h^2}{12}\mu + h^4 L^h\rho^h + 4h^2 w_{01} \\ &+h^4 dw_{01} - 2h^2 w_{00} - h^4\mu^h_{1x} - h^4\mu^h_{1y} - h^4\mu^h_{00}. \end{aligned}$$

Again, on the basis of equations (1.1), (1.10) and definitions (3.5) cancellation of terms of the orders 1 and h^2 takes place. Finally, after division by h^4 we arrive at the equality

$$L^h \rho^h = -dw_{01} + \mu^h_{1x} + \mu^h_{1y} + \mu^h_{00} \quad \text{in} \quad \omega_{10}. \tag{3.20}$$

Similarly for the nodes of another group of alternating evenness ω_{01} we obtain the equality

$$L^h \rho^h = -dw_{01} + \mu^h_{1x} + \mu^h_{1y} + \mu^h_{00} \quad \text{in} \quad \omega_{01} \tag{3.21}$$

with the same estimate (3.19) for the remainder term μ^h_{00}.

Taking into consideration (3.3), (3.4), (3.12), (3.16), (3.20), and (3.21), for ρ^h we obtain the problem

$$\begin{aligned} &L^h \rho^h = \xi^h \quad \text{in} \quad \omega_h \setminus \omega_{00}, \\ &L^h \rho^h - L^{2h} \rho^h = \xi^h \quad \text{in} \quad \omega_{00}, \\ &\rho^h = -\omega_{01} \quad \text{in} \quad \gamma_h \cap (\overline{\omega}_{01} \cup \overline{\omega}_{10}), \\ &\rho^h = -\omega_{00} \quad \text{in} \quad \gamma_h \cap \overline{\omega}_{00} \end{aligned} \tag{3.22}$$

with the right-hand side

$$\begin{aligned} &\xi^h = \mu^h_{1x} + \mu^h_{1y} + 2\mu^h_{01} \quad \text{in} \quad \omega_{11}, \\ &\xi^h = -dw_{01} + \mu^h_{1x} + \mu^h_{1y} + \mu^h_{00} \quad \text{in} \quad \omega_{01} \cup \omega_{10}, \\ &\xi^h = \mu^h_{1x} + \mu^h_{1y} - \mu^h_{2x} - \mu^h_{2y} - dw_{00} \quad \text{in} \quad \omega_{00}. \end{aligned}$$

Owing to the estimates (3.8), (3.11), (3.15), (3.19) and boundedness of functions d and μ from (3.5) the following inequality is valid

$$\left|\xi^h\right| \le c_7 \quad \text{in} \quad \omega_h. \tag{3.23}$$

Use the a priori estimate from Theorem 3. Then with account of (3.5) we have

$$\left\|\rho^h\right\|_{\infty,\overline{\omega}_h} \le \frac{11}{48}\left\|\xi^h\right\|_{\infty,\omega_h} + \frac{1}{12}\|\mu\|_{\infty,\gamma_h}. \tag{3.24}$$

Taking into account the estimate (3.23), we obtain (3.6) with the constant

$$c_6 = 11c_7/48 + \|\mu\|_{\infty,\gamma_h}/12.$$

From the representation (3.2) — (3.4) it follows that

$$\left\|u - u^h\right\|_{\infty,\overline{\omega}_h} \le h^4 \left(\left\|\rho^h\right\|_{\infty,\overline{\omega}_h} + \|w_{01}\|_{\infty,\overline{\Omega}} + \|w_{00}\|_{\infty,\overline{\Omega}}\right).$$

With account of (3.24) this proves the estimate (3.1). □

4 Numerical examples

By analogy with the work [1] apply the constructed method to two problems of the form (1.1)–(1.2) with smooth and with oscillation solutions. The first problem is

$$
\begin{aligned}
-\Delta u = \; & 2\cos\left(\frac{\pi x}{2}\right) y(1-y)\cos\left(\frac{\pi y}{2}\right) \\
& +(1-x)\sin\left(\frac{\pi x}{2}\right)\pi y(1-y)\cos\left(\frac{\pi y}{2}\right) \\
& -x\sin\left(\frac{\pi x}{2}\right)\pi y(1-y)\cos\left(\frac{\pi y}{2}\right) \\
& +\frac{1}{2}x(1-x)\cos\left(\frac{\pi x}{2}\right)\pi^2 y(1-y)\cos\left(\frac{\pi y}{2}\right) \\
& +2x(1-x)\cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right) \qquad (4.1) \\
& +x(1-x)\cos\left(\frac{\pi x}{2}\right)(1-y)\sin\left(\frac{\pi y}{2}\right)\pi \\
& -x(1-x)\cos\left(\frac{\pi x}{2}\right) y\sin\left(\frac{\pi y}{2}\right)\pi \quad \text{in} \quad \Omega, \\
u = \; & 0 \quad \text{in} \quad \Gamma.
\end{aligned}
$$

Its exact solution is

$$u(x,y) = x(1-x)\cos\left(\frac{\pi x}{2}\right) y(1-y)\cos\left(\frac{\pi y}{2}\right).$$

The second problem is

$$
\begin{aligned}
-\Delta u = \; & -32c(1-x)y(1-y) + 512sx(1-x)y(1-y) \\
& +32cxy(1-y) + 2sy(1-y) - 32cx(1-x)(1-y) \\
& +32cx(1-x)y + 2sx(1-x) \quad \text{in} \quad \Omega, \qquad (4.2) \\
u = \; & 0 \quad \text{in} \quad \Gamma,
\end{aligned}
$$

where the denotations $s = \sin(16x+16y)$ and $c = \cos(16x+16y)$ are used. Its exact solution is

$$u(x,y) = \sin(16x+16y)x(1-x)y(1-y).$$

In Tables 1,2 the errors $\delta_2 = \|u - u^h\|_{\infty,\overline{\omega}_h}$ and

$$\delta_1 = \|u - u^h\|_{2,\overline{\omega}_h} = \left(\sum_{z\in\overline{\omega}_h} \left(u(z) - u^h(z)\right)^2\right)^{1/2}$$

of solutions of both the problems by standard method (1.7) of the second order of accuracy and by the proposed method (1.10)–(1.12) of the fourth order are presented.

Table 1: Error of approximate solutions for the problem with smooth solution.

N	Problem I			
	method (1.7)		method (1.10) — (1.12)	
	$2, \overline{\omega}_h$	$\infty, \overline{\omega}_h$	$2, \overline{\omega}_h$	$\infty, \overline{\omega}_h$
4	$1.18_{10}-03$	$2.24_{10}-03$	$6.73_{10}-04$	$2.20_{10}-03$
8	$2.92_{10}-04$	$6.11_{10}-04$	$4.30_{10}-05$	$1.64_{10}-04$
16	$7.27_{10}-05$	$1.52_{10}-04$	$2.68_{10}-06$	$1.02_{10}-05$
32	$1.82_{10}-05$	$3.82_{10}-05$	$1.68_{10}-07$	$6.46_{10}-07$
64	$4.54_{10}-06$	$9.54_{10}-06$	$1.06_{10}-08$	$4.08_{10}-08$

Table 2: Error of approximate solutions for the problem with oscillating solution.

N	Problem II			
	method (1.7)		method (1.10) — (1.12)	
	$2, \overline{\omega}_h$	$\infty, \overline{\omega}_h$	$2, \overline{\omega}_h$	$\infty, \overline{\omega}_h$
4	$1.38_{10}-01$	$2.70_{10}-01$	$1.53_{10}-01$	$3.64_{10}-01$
8	$1.18_{10}-02$	$2.67_{10}-02$	$4.70_{10}-02$	$1.41_{10}-01$
16	$2.42_{10}-03$	$5.76_{10}-03$	$2.57_{10}-03$	$1.04_{10}-02$
32	$5.78_{10}-04$	$1.39_{10}-03$	$1.42_{10}-04$	$6.00_{10}-04$
64	$1.43_{10}-04$	$3.52_{10}-04$	$8.45_{10}-06$	$3.61_{10}-05$

These data are represented on graphs (in logarithmic scale over both axes). In figures 4 and 5 the errors δ_1 and δ_2 of the method (1.7) are denoted by numbers 1, 2; The errors of the method (1.10) — (1.12) are denoted by numbers 3, 4; numbers 5 and 6 denote the lines with inclinations $tg(\varphi) = 2$ and $tg(\varphi) = 4$, characterizing the dependences $\delta = h^2$ and $\delta = h^4$, respectively.

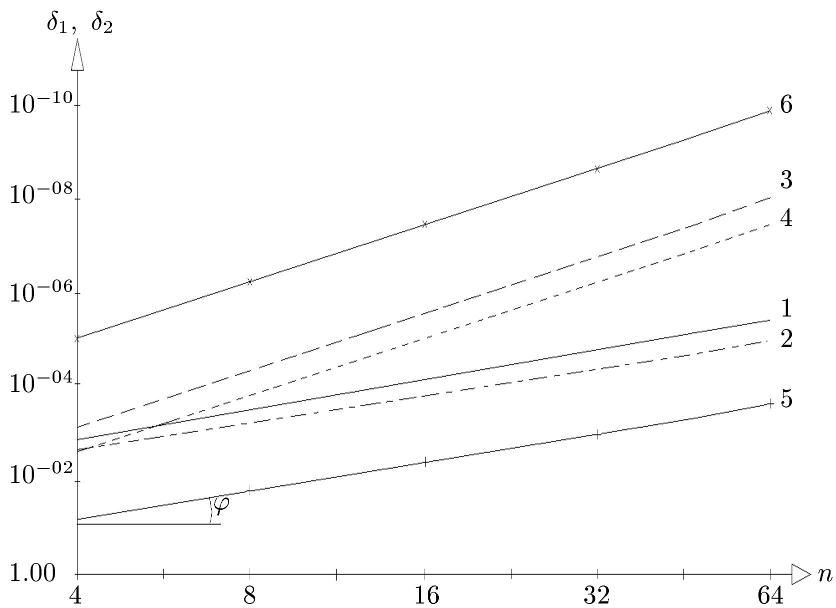

Fig. 4: Error of approximate solutions for the first problem.

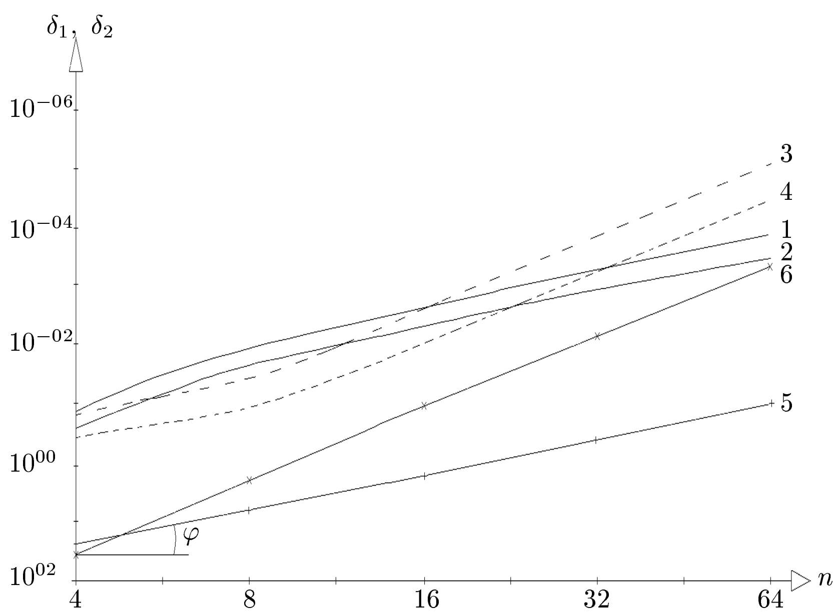

Fig. 5: Error of approximate solutions for the second problem.

Except that, in Fig. 6 a pointwise graph of the error $\delta_2' = u - u^h$ of the proposed method (1.10)–(1.12) on a grid $\overline{\omega}_h$ with the step $h = 1/32$ for the first problem is given.

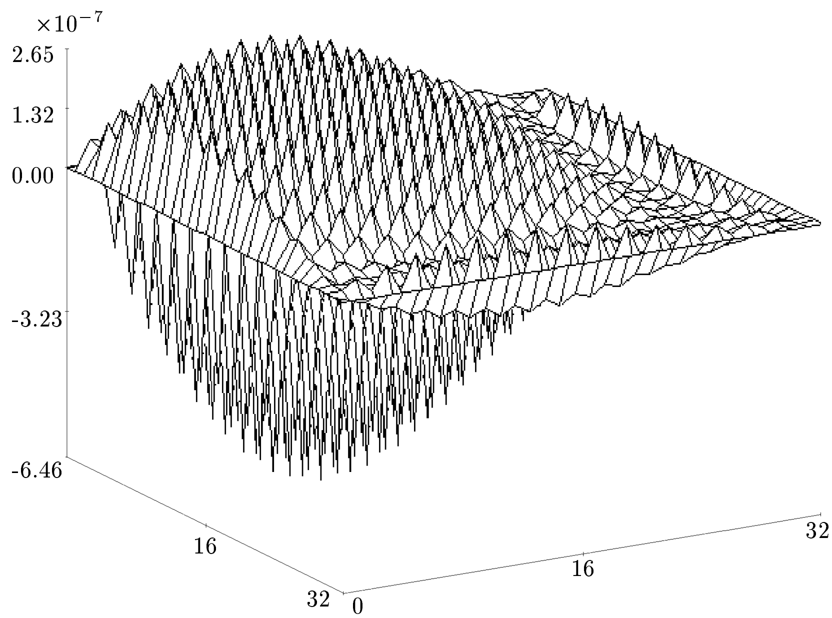

Fig. 6: Error δ_2' of the method (1.10)–(1.12) under $n = 32$. The first problem.

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