Experimental analysis of fourth-order schemes for Poisson's equation

Bykova E.G., Rüde U., Shaidurov V.V

Introduction

This is not the first attempt to perform a comparison of numerical schemes for Poisson's equation [3]. However, during the last few years some new approach had been developed which was not studied experimentally in a comparison. Here, we consider several finite-difference schemes for Poisson's equation with Dirichlet boundary condition and evaluate them for three different types of solution: smooth, oscillatory and exponentially growing. The results are evaluated in the discrete L_2-, L_∞- and energy norms. In all computations, the problem is discretized on uniform square mesh (divided into triangles, if necessary). Of course, a uniform mesh does not permit to demonstrate the ability of some methods to adapt for an arbitrary (triangle or quadrangle) meshes. Moreover, different methods on a uniform mesh may result in same discrete algebraic systems if they are combined with appropriate quadrature rules for the right-hand side. Nevertheless, even these simple comparisons yield interesting insights.

1 Formulation of the differential problems

Let $\Omega = (0,1) \times (0,1)$ be the unit square with the boundary Γ. Consider the Dirichlet problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \tag{I}$$
$$u = 0 \quad \text{on} \quad \Gamma. \tag{II}$$

We shall treat three examples with known exact solution. (The first and second examples are taken from [3]).

Example 1. Let

$$\begin{aligned} f(x,y) := f_1(x,y) &= c_x c_y (2y(1-y) + 2x(1-x) + \pi^2 x(1-x)y(1-y)/2) \\ &+ s_x c_y \pi(1-2x)y(1-y) + c_x s_y \pi(1-2y)x(1-x) \end{aligned} \tag{III}$$

where

$$\begin{aligned} s_x &= \sin(\pi x/2),\ c_x = \cos(\pi x/2), \\ s_y &= \sin(\pi y/2),\ c_y = \cos(\pi y/2). \end{aligned}$$

This right-hand side gives rise to a comparatively smooth solution of problem (I)–(II):

$$u(x,y) := u_1(x,y) = x(1-x)\cos(\pi x/2)y(1-y)\cos(\pi y/2). \tag{IV}$$

Example 2. Let

$$\begin{aligned} f(x,y) := f_2(x,y) &= -32c(1-2x)y(1-y) \\ &+ 512sx(1-x)y(1-y) + 2sy(1-y) \\ &- 32cx(1-x)(1-2y) + 2sx(1-x) \end{aligned} \tag{V}$$

where

$$s = \sin(16x + 16y),\ c = \cos(16x + 16y).$$

With this right-hand side we obtain an oscillatory solution of problem (I)–(II):

$$u(x,y) := u_2(x,y) = \sin(16x + 16y)x(1-x)y(1-y). \tag{VI}$$

Example 3. Let

$$\begin{aligned} f(x,y) := f_3(x,y) &= (x(1-x)y(y+3) \\ &+ x(x+3)y(1-y))e^{x+y}. \end{aligned} \tag{VII}$$

For this right-hand side we obtain an exponentially growing but comparatively smooth solution of problem (I)–(II):

$$u(x,y) := u_3(x,y) = x(1-x)y(1-y)e^{x+y}. \tag{VIII}$$

2 Tested methods

Let

$$\bar{\Omega}_h = \{z_{ij} : z_{ij} = (x_i, y_j);\ x_i = ih,\ i = 0, 1, \ldots, n;\ y_j = jh,\ j = 0, 1, \ldots, n\}$$

be uniform square grid with mesh-size $h = 1/n$. Let also

$$\Omega_h = \{z_{ij} : z_{ij} \in \bar{\Omega}_h \bigcap \Omega\}$$

and

$$\Gamma_h = \{z_{ij} : z_{ij} \in \bar{\Omega}_h \bigcap \Gamma\}.$$

To simplify the notation, we shall use the shortening

$$v_{ij} = v(z_{ij}) = v(x_i, y_j).$$

2.1 Five-point scheme and Richardson extrapolation

Here we use standard scheme

$$\begin{aligned} &\frac{4}{h^2} u^h_{i,j} - \frac{1}{h^2} u^h_{i+1,j} - \frac{1}{h^2} u^h_{i-1,j} - \frac{1}{h^2} u^h_{i,j+1} - \frac{1}{h^2} u^h_{i,j-1} = f_{ij}, && \text{(I)} \\ &i, j = 1, \ldots, n-1, \quad \text{i.e.,} \quad z_{ij} \in \Omega_h; && \\ &u^h_{ij} = 0 \quad \text{if} \quad z_{ij} \in \Gamma_h. && \text{(II)} \end{aligned}$$

Of course, the solution of this problem has only second order of accuracy. However, using Richardson extrapolation the accuracy can be improved. For this purpose we assume n to be even and solve one more auxiliary problem (I)–(II) with mesh-size $2h$. Then we take both solutions u^h and u^{2h} and form a linear combination

$$u^{Rich}(z) = \frac{4}{3} u^h(z) - \frac{1}{3} u^{2h}(z) \quad \forall z \in \bar{\Omega}_{2h}. \qquad \text{(III)}$$

According to the theory, this combination has fourth order of accuracy in the discrete L_∞-norm [4].

2.2 Nonhomogeneous Bykova-Shaidurov scheme

This discretization uses different stencils at different grid points [5], [6]. Let again n be even. In the nodes (i, j) with both i and j even, this scheme has the form

$$\begin{aligned}
&\frac{3}{h^2}u^h_{i,j} - \frac{1}{h^2}u^h_{i+1,j} - \frac{1}{h^2}u^h_{i-1,j} - \frac{1}{h^2}u^h_{i,j+1} - \frac{1}{h^2}u^h_{i,j-1} \\
&+\frac{1}{4h^2}u^h_{i+2,j} + \frac{1}{4h^2}u^h_{i-2,j} + \frac{1}{4h^2}u^h_{i,j+2} + \frac{1}{4h^2}u^h_{i,j-2} = 0, \qquad \text{(IV)} \\
&i,j = 2,4,\ldots,n-2.
\end{aligned}$$

At the rest nodes of Ω_h we use equations (I) and, finally, on the boundary nodes Γ_h we use equation (II). In [5] the fourth order of accuracy is proved in discrete L_∞-norm.

2.3 Khoromskij combination

The method is similar to Richardson extrapolation and uses solutions of two difference schemes [7]. But this time we perform the computation on the same grid and n is not necessarily even. The first scheme coincides with (I)–(II). The second one uses the oblique 5-point cross:

$$\begin{aligned}
&\frac{1}{2h^2}(4\bar{u}^h_{i,j} - \bar{u}^h_{i+1,j+1} - \bar{u}^h_{i-1,j-1} - \bar{u}^h_{i-1,j+1} - \bar{u}^h_{i+1,j-1}) = f_{ij}, \qquad \text{(V)} \\
&i,j = 1,\ldots,n-1; \\
&\bar{u}^h_{i,j} = 0 \quad \text{if} \quad z_{ij} \in \Gamma_h. \qquad \text{(VI)}
\end{aligned}$$

Then we form the linear combination

$$u^{Khor}(z) = \frac{2}{3}u^h(z) + \frac{1}{3}\bar{u}^h(z) - \frac{h^2}{12}f(z) \quad \forall z \in \bar{\Omega}_h. \qquad \text{(VII)}$$

According to the proof in [7], this combination has fourth order of accuracy in discrete L_∞-norm.

2.4 Nine-point box scheme

This scheme uses only one grid and is homogeneous in the sense that it exploits only one 9-point stencil over all inner nodes of the grid [1], [2]. We apply it in the following form:

$$\begin{aligned}
&\frac{1}{6h^2}(20u^h_{i,j} - 4u^h_{i+1,j} - 4u^h_{i-1,j} - 4u^h_{i,j+1} - 4u^h_{i,j-1} \\
&-u^h_{i+1,j+1} - u^h_{i-1,j-1} - u^h_{i-1,j+1} - u^h_{i+1,j-1}) = f_{i,j} + \frac{h^2}{12}(\Delta f)_{i,j}, \qquad \text{(VIII)} \\
&i,j = 1,\ldots,n-1.
\end{aligned}$$

For the boundary nodes Γ_h we again use equations (II). Note, that right-hand side in (VIII) is often used in the form

$$\frac{2}{3}f_{i,j} + \frac{1}{12}f_{i+1,j} + \frac{1}{12}f_{i-1,j} + \frac{1}{12}f_{i,j+1} + \frac{1}{12}f_{i,j-1}.$$

The difference between them is of fourth order of smallness and therefore they both give the same fourth order of accuracy for the difference solution in the discrete L_∞-norm [1], [2]. From practical point of view, the last value is preferable since does not involve an analytical modification of the right-hand side. But it contains difference differentiation in an implicit form. In order to eliminate the additional truncation error, we have used (VIII) with the exact analytical differentiation in all our problems.

3 Two ways to compare the computational cost

The traditional basis for a comparison is simply to use the number of unknowns as a measure of complexity. So we simply use the same grids with number of inner nodes $(n-1)^2$ for all example problems. Therefore, we performed the computation for $n = 2, 4, 8, 16, 32, 64$ and display the results for Example 1 in fig. 1 (top), 2 (top), and 3 (top) which correspond to the evaluated discrete energy-, L_∞-, and L_2-norms, respectively. The figures plot the error versus the number of mesh points. In each figure

line 1 (marked by asterisks) demonstrates	Richardson extrapolation,
line 2 (marked by dots) corresponds to	Bykova-Shaidurov scheme,
line 3 (marked by crosses) demonstrates	Khoromskij combination,
line 4 (marked by circles) corresponds to	nine-point box scheme.

The second comparison is based on the number of non-zero coefficients of the system matrices. This number is the amount of input data for the iterative process and should be useful for the evaluation of the complexity of smoother iterations (s.f. [8]). This point of view changes the situation, since the different methods on the same $(n-1) \times (n-1)$ grid result in the following number of coefficients:

in Richardson extrapolation	$6.25n^2$,
in Shaidurov-Bykova scheme	$6n^2$,
in Khoromskij combination	$10n^2$,
in nine-point box scheme	$9n^2$.

The result for this approach are displayed for Example 1 in figures 1 (bottom), 2 (bottom), and 3 (bottom) for the discrete energy-, L_∞-, and L_2-norms, respectively.

From the figures 1, 2, 3 one can see that the difference between first and second comparison criteria of is not significant for the relative ranking

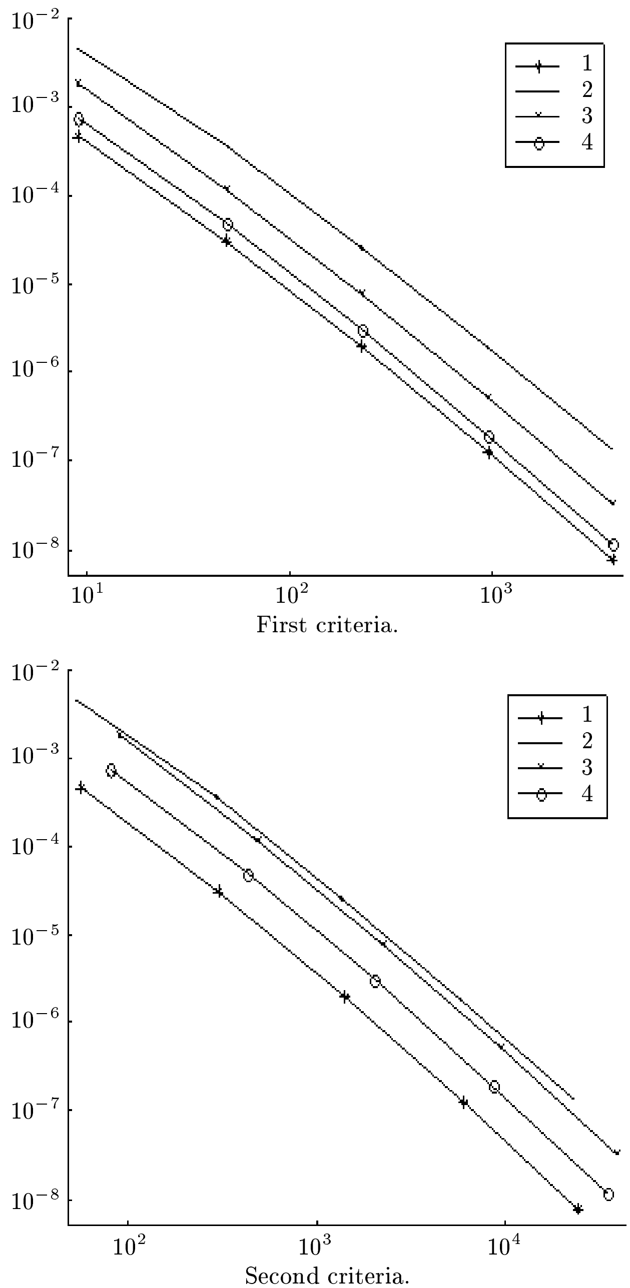


Fig. 1. Energy-norm of error in Example 1.

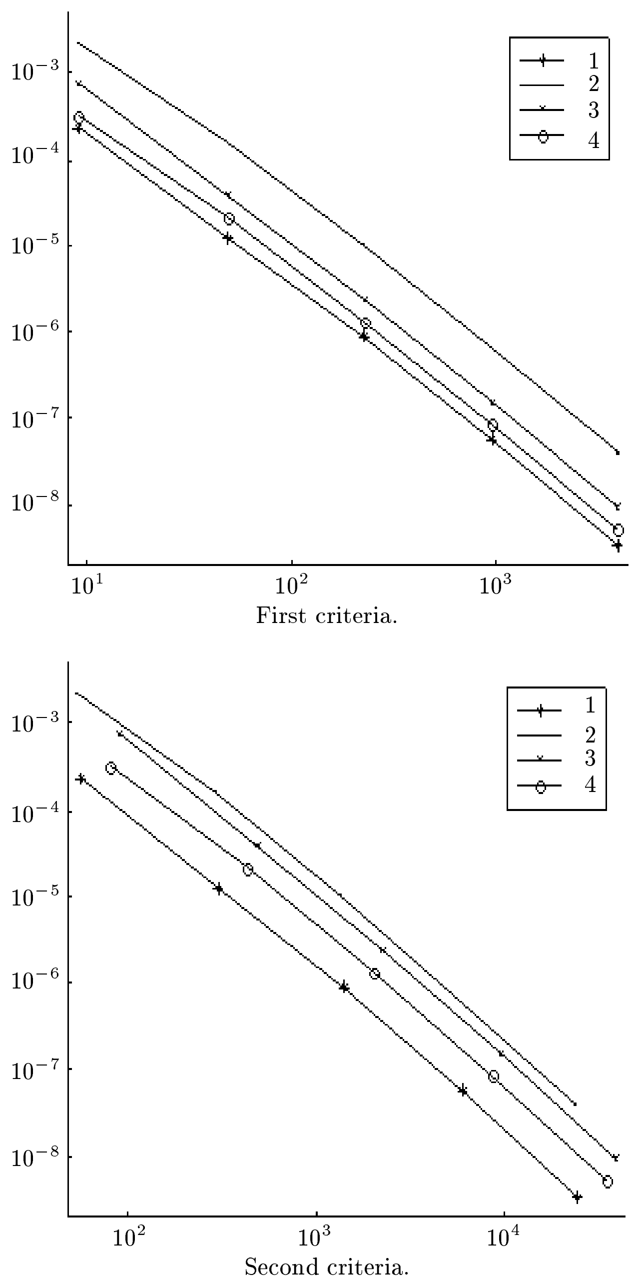
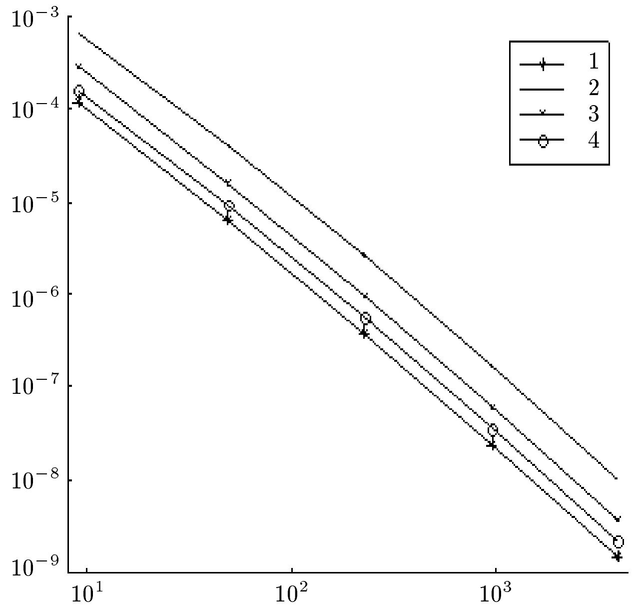
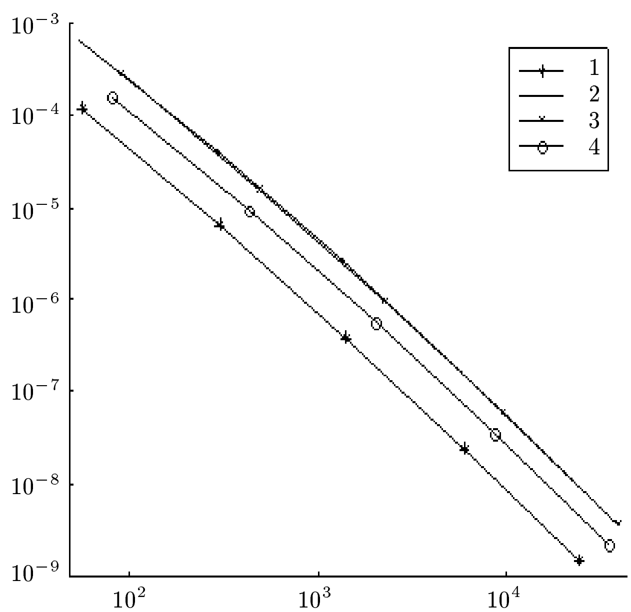


Fig. 2. L_∞−norm of error in Example 1.

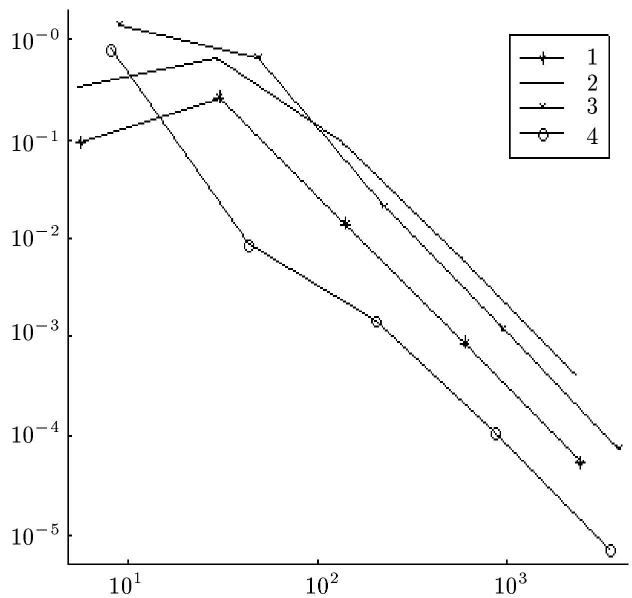


First criteria.

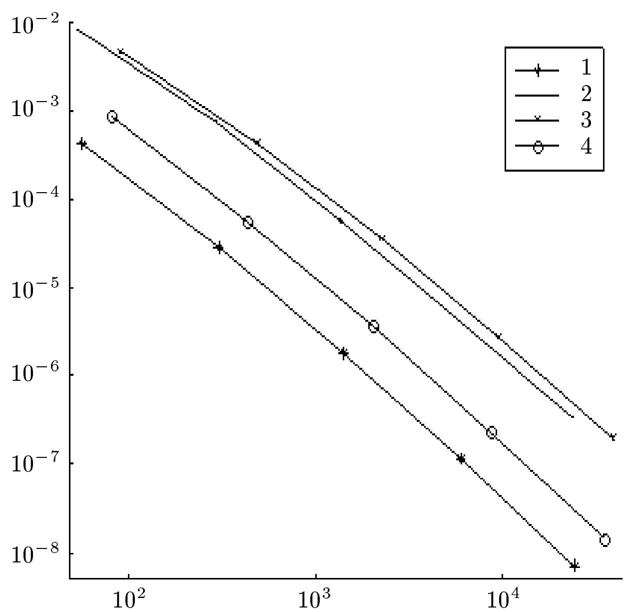


Second criteria.

Fig. 3. L_2-norm of error in Example 1.

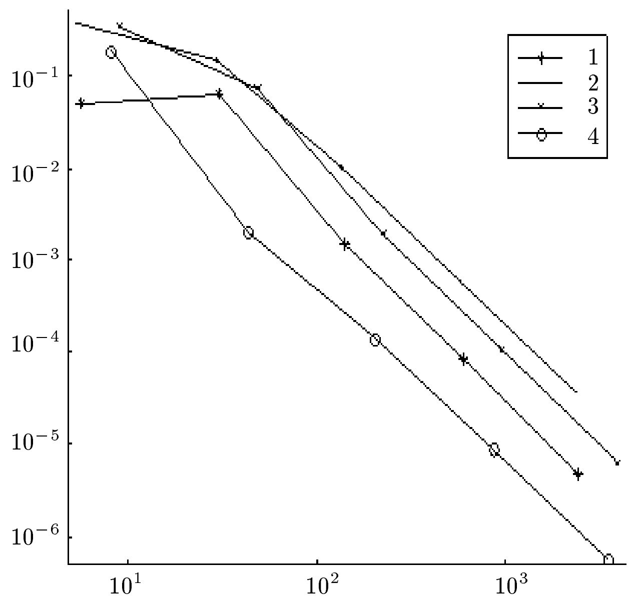


Example 2.

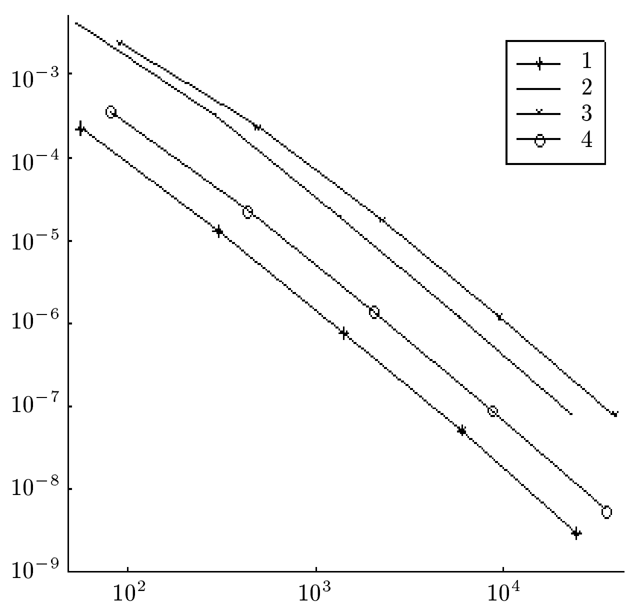


Example 3.

Fig. 4. Energy-norm of error for Examles 2 and 3. Second criterium.

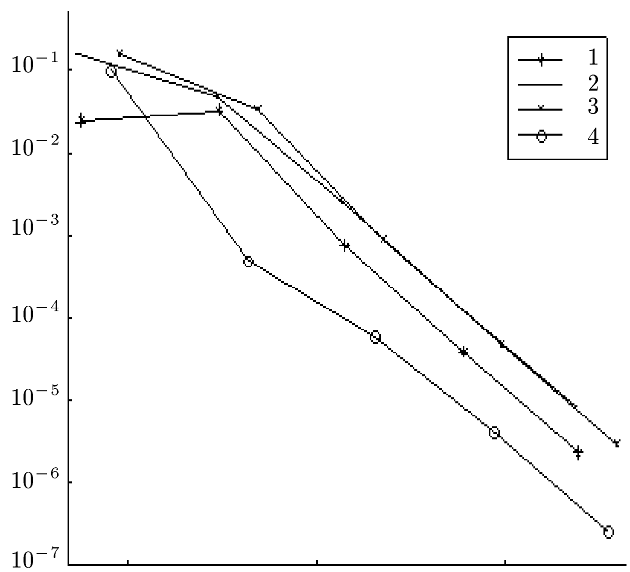


Example 2.

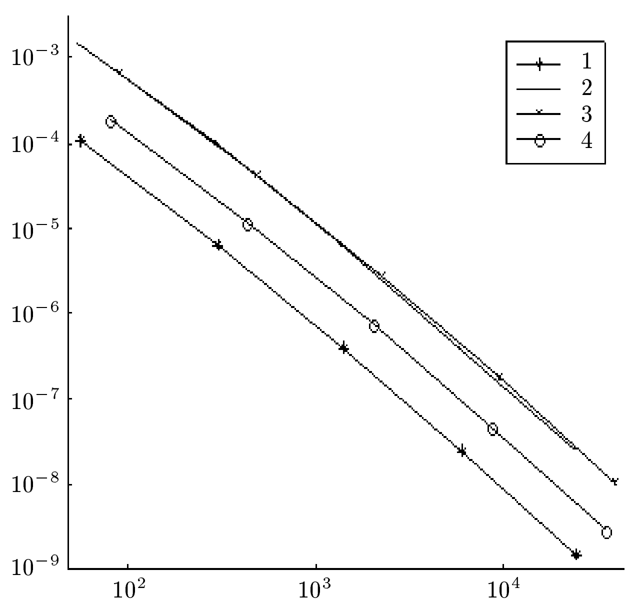


Example 3.

Fig. 5. L_∞−norm of error for Examples 2 and 3. Second criterium.



Example 2.



Example 3.

Fig. 6. L_2-norm of error for Examples 2 and 3. Second criterium.

of the methods. Therefore in Example 2 and 3 we present only the results for the second type of comparison where the complexity is evaluated with respect to number of nonzero coefficients of the matrices. In figures 4, 5, and 6 we show graphs of errors for both examples in the discrete energy norm, L_∞- and L_2-norms, respectively. In each figure the graphs for the L_∞- and L_2-norms are asymptotically lines with a slope that clearly indicates an $O(h^4)$-behavior.

Summarizing, in Examples 1 and 3, where the solution is smooth, Richardson extrapolation is most effective in all norms among the tested methods. For the oscillatitive solution of Example 2, the nine-point box scheme is most efficient, again in all three norms.

References

1. Collatz L.: *The Numerical Treatment of Differential Equations.* Berlin, 1966.
2. Samarskij A.A.: *Theorie der Differenzenverfahren.* Leipzig, 1984.
3. Rüde U.: *Extrapolation and Related Techniques for Solving Elliptic Equations.* Preprint № I-9135, München Technical University, 1991.
4. Marchuk G.I., Shaidurov V.V.: *Difference Methods and Their Extrapolations.* N.Y., Springer, 1983.
5. Bykova E.G., Shaidurov V.V.: *A two-dimensional nonuniform difference scheme with higher order of accuracy.* This book.
6. Bykova E.G., Shaidurov V.V.: *A nonuniform difference scheme with fourth order of accuracy in a domain with smooth boundary.* This book.
7. Khoromskij B.N.: *Method of increasing accuracy of discrete solutions of boundary-value problems with the operator invariant with respect to turning of coordinate system.* Preprint № P5-80-736, Institute of Nuclear resourches, Dubna, 1980 (in Russian).
8. Hackbusch W.: *Theorie und Numerik elliptischer Differentialgleichungen.* Stuttgart, 1996.