Completely splitting method for the Navier-Stokes problem

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Introduction

In this part we consider two-dimensional time-dependent Navier-Stokes equations in a rectangular domain and study the method of full splitting [4]-[7]. On the physical level, this problem is splitted into two processes: convection-diffusion and work of pressure. The convection-diffusion step is further splitted in two geometric directions. To implement the finite element method, we use the approach with uniform square grids which are staggered relative to one another. This allows the Ladyzhenskaya-Babuška-Brezzi condition for stability of pressure to be fulfilled without usual diminishing the number of degrees of freedom for pressure relative to that for velocities. For pressure we take piecewise constant finite elements. As for velocities, we use piecewise bilinear elements.

1 The formulation of the problem and the splitting into physical processes

In the rectangular domain $\Omega = (0,1) \times (0,1)$ with the boundary Γ we consider the two-dimensional Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{Re}\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega \times (0,T), \tag{1.1}$$

the continuity equation

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{1.2}$$

the boundary condition

$$\mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma \times [0, T], \tag{1.3}$$

and the initial condition

$$\mathbf{u}(x, y, 0) = \mathbf{u}_0(x, y) \quad \text{on} \quad \Omega. \tag{1.4}$$

Here $\mathbf{u}(x,y,t) = (u_1(x,y,t), u_2(x,y,t))$ is an unknown speed vector-function; $p(x,y,t)$ is an unknown function; $\mathbf{f}(x,y,t) = (f_1(x,y,t), f_2(x,y,t))$ is a given vector-function; $\mathbf{g}(x,y,t) = (g_1(x,y,t), g_2(x,y,t))$ is a given continuous vector-function on $\Gamma \times [0,1]$; $\mathbf{u}_0(x,y) = (u_{0,1}(x,y), u_{0,2}(x,y))$ is a given continuous vector-function on Ω; Re is the Reynolds number.

If these equations have a solution $\mathbf{u}, p$ then one can see that a pair $\mathbf{u}, p+c$ is also a solution for any constant c. In order to exclude the multivalence we demand that

$$\int_\Omega p \, d\Omega = 0. \tag{1.5}$$

Rewrite the vector equation (1.1) in the form of two scalar ones. Put $\nu = 1/Re$ and replace the third term of (1.1) by equivalent sum of two expressions on account of the continuity equation:

$$\frac{\partial u_1}{\partial t} - \nu \Delta u_1 + \frac{1}{2}(\mathbf{u} \cdot \nabla) u_1 + \frac{1}{2} div(u_1 \mathbf{u}) + \frac{\partial p}{\partial x} = f_1, \tag{1.6}$$

$$\frac{\partial u_2}{\partial t} - \nu \Delta u_2 + \frac{1}{2}(\mathbf{u} \cdot \nabla) u_2 + \frac{1}{2} div(u_2 \mathbf{u}) + \frac{\partial p}{\partial y} = f_2. \tag{1.7}$$

For the obtained problem (1.2)–(1.7), at first we consider Chorin's splitting method [4] - [7] (of fractional steps) into two physical processes: transfer with diffusion of substance and pressure work. Therefore, the time interval $[0, T]$ is divided into m equal segments, $\tau = T/m$ long, by the nodes of the time grid

$$\bar{\omega}^\tau = \{t_k : t_k = k\tau, \quad k = 0, 1, \ldots, m\}.$$

Let us introduce also

$$\omega^\tau = \bar{\omega}^\tau \setminus \{0\}.$$

Instead of the exact functions $\mathbf{u}$ and p we will seek a vector-function $\mathbf{u}_k^\tau(x,y) = (u_{1,k}^\tau(x,y), u_{2,k}^\tau(x,y))$ and a function $p_k^\tau(x,y)$ which are determined at a discrete instant of time $t = k\tau$.

At first we use the condition (1.4) and put

$$\mathbf{u}_0^\tau(x, y) = \mathbf{u}_0(x, y) \quad \text{in} \quad \Omega. \tag{1.8}$$

Then we construct the sequence of problems alternating on every segment $[t_k, t_{k+1}]$. Two first problems for $s = 1$ and $s = 2$ are not connected with each other and are required to determine the vector-function $\mathbf{v}(x, y, t) = (v_1(x, y, t), v_2(x, y, t))$:

$$\frac{\partial v_s}{\partial t} - \nu \Delta v_s + \frac{1}{2}(\mathbf{u}_k^\tau \cdot \nabla) v_s + \frac{1}{2} div(v_s \mathbf{u}_k^\tau) = \frac{1}{2} f_s \quad \text{in} \quad \Omega \times (t_k, t_{k+1}), \tag{1.9}$$

$$v_s = g_s \quad \text{on} \quad \Gamma \times [t_k, t_{k+1}], \tag{1.10}$$

$$v_s(x, y, t_k) = u_{s,k}^\tau(x, y) \quad \text{in} \quad \Omega. \tag{1.11}$$

After this the obtained function at time level t_{k+1} is used as an initial value for the other problem for the determination of the vector-function $\mathbf{w}(x, y, t_k) = (w_1(x, y, t_k), w_2(x, y, t_k))$ and the function $q(x, y, t_k)$ on the *same* segment $[t_k, t_{k+1}]$:

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla q = \frac{1}{2}\mathbf{f} \quad \text{in} \quad \Omega \times (t_k, t_{k+1}), \tag{1.12}$$

$$div\, \mathbf{w} = 0 \quad \text{in} \quad \Omega \times (t_k, t_{k+1}), \tag{1.13}$$

$$\mathbf{w} \cdot \mathbf{n} = \mathbf{g} \cdot \mathbf{n} \quad \text{on} \quad \Gamma \times (t_k, t_{k+1}), \tag{1.14}$$

$$\mathbf{w}(x, y, t_k) = \mathbf{v}(x, y, t_{k+1}) \quad \text{in} \quad \Omega, \tag{1.15}$$

where $\mathbf{n}(x, y)$ is the vector of outer normal to the boundary Γ at a point $(x, y) \in \Gamma$, which is redefined at a vertex of a square.

The solution of the splitting problem at the time point t_{k+1} is a result of a loop on the segment $[t_k, t_{k+1}]$:

$$\mathbf{u}_{k+1}^\tau(x, y) = \mathbf{w}(x, y, t_{k+1}), \tag{1.16}$$

$$p_{k+1}^\tau(x, y) = q(x, y, t_{k+1}) \quad \text{in} \quad \Omega. \tag{1.17}$$

Repeating this computation loop for $k = 0, \ldots, m-1$, we sequentially obtain the values of functions $\mathbf{u}^\tau$ and p^τ at time levels $\tau, \ldots, T$.

Remark 1. It is necessary to pay attention to the change of the boundary condition (1.14) in comparison with (1.3). This condition follows from (1.3), but it is less limiting. The substitution is necessary because the condition

$$\mathbf{w} = \mathbf{g} \quad \text{on} \quad \Gamma \times [t_k, t_{k+1}]$$

gives an overdetermined problem. □

2 Discretization of the fractional step of pressure work

Consider the problem (1.12)–(1.15) and sequentially carry out the time discretization and then the space one. The time discretization is realized by the replacement the derivative $\partial/\partial t$ with the difference ration:

$$\frac{\partial w_1}{\partial t}(x,y,t) \approx (w_1(x,y,t) - w_1(x,y,t-\tau))/\tau. \tag{2.1}$$

After rearranging the known terms to the right-hand side we obtain the stationary differential problem at time level t_{k+1}:

$$\frac{1}{\tau}w_1^{k+1} + \frac{\partial q^{k+1}}{\partial x} = \frac{1}{2}f_1^{k+1} + \frac{1}{\tau}w_1^k \quad \text{in} \quad \Omega, \tag{2.2}$$

$$\frac{1}{\tau}w_2^{k+1} + \frac{\partial q^{k+1}}{\partial y} = \frac{1}{2}f_2^{k+1} + \frac{1}{\tau}w_2^k \quad \text{in} \quad \Omega, \tag{2.3}$$

$$\frac{\partial w_1^{k+1}}{\partial x} + \frac{\partial w_2^{k+1}}{\partial y} = 0 \quad \text{in} \quad \Omega \tag{2.4}$$

with the boundary condition

$$\mathbf{w}^{k+1}\cdot\mathbf{n} = \mathbf{g}^{k+1}\cdot\mathbf{n} \quad \text{on} \quad \Gamma. \tag{2.5}$$

From here on for an arbitrary function the notation u^k means $u(t_k)$.

For the space discretization we apply the finite element method. Therefore turn to the generalized formulation. Consider three arbitrary functions $v_1(x,y)$, $v_2(x,y)$, $r(x,y)$; two ones satisfy the boundary condition

$$v_1 n_1 + v_2 n_2 = 0 \quad \text{on} \quad \Gamma. \tag{2.6}$$

Multiply the equations (2.2)–(2.4) by v_1, v_2, q respectively, combine them, integrate by parts over Ω, and apply the condition (2.6). As a result, we obtain

$$\begin{aligned}
&\frac{1}{\tau}(w_1^{k+1}, v_1)_\Omega + \frac{1}{\tau}(w_2^{k+1}, v_2)_\Omega - \left(q^{k+1}, \frac{\partial v_1}{\partial x}\right)_\Omega \\
&- \left(q^{k+1}, \frac{\partial v_2}{\partial y}\right)_\Omega + \left(\frac{\partial w_1^{k+1}}{\partial x}, r\right)_\Omega + \left(\frac{\partial w_2^{k+1}}{\partial y}, r\right)_\Omega \\
&= \frac{1}{2}(f_1^{k+1}, v_1)_\Omega + \frac{1}{2}(f_2^{k+1}, v_2)_\Omega + \frac{1}{\tau}(w_1^k, v_1)_\Omega + \frac{1}{\tau}(w_2^k, v_2)_\Omega
\end{aligned} \tag{2.7}$$

where $(\cdot,\cdot)_\Omega$ means the scalar product

$$(u,v) = \int_\Omega uv\, d\Omega.$$

In this paper from time to time we shall use a method of fictitious domains in the small (near the boundary). First, let us introduce the domain $\Omega_1 = (0,1)\times(-h/2, 1+h/2)$ and divide it into $n(n+1)$ squares

$$e_{i+1/2,j} = (x_i, x_{i+1}) \times (y_{j-1/2}, y_{j+1/2})$$

by lines

$$\begin{aligned} x_i &= ih, \quad i = 0,\ldots,n; \\ y_{j+1/2} &= (j+1/2)h, \quad j = -1,\ldots,n. \end{aligned}$$

For v_1, w_1^{k+1} we introduce the space H_x of admissible functions which are continuous on $\bar\Omega_1$ and bilinear on each $e_{i+1/2,j} \subset \Omega_1$. The degrees of freedom of these functions are referred to the nodes $z_{i,j+1/2} = (x_i, y_{j+1/2})$. We denote the set of these nodes

$$\bar\Omega_1^h = \{z_{i,j+1/2} : i = 0,\ldots,n,\ j = -1,\ldots,n\}$$

and introduce its inner part

$$\Omega_1^h = \bar\Omega_1^h \cap \Omega.$$

Then as the basis function corresponding to the node $z_{i,j+1/2}$ we take $\varphi_{x,i,j+1/2} \in H_x$ which equals 1 at $z_{i,j+1/2}$ and 0 at any other node of $\bar\Omega_1^h$.

The arrangement of nodes $\bar\Omega_1^h$ and some basis functions from H_x are represented in Fig. 1, 2.

Second, let us introduce the domain $\Omega_2 = (-h/2, 1+h/2)\times(0,1)$ and divide it into $n(n+1)$ squares

$$e_{i,j+1/2} = (x_{i-1/2}, x_{i+1/2}) \times (y_j, y_{j+1})$$

by lines

$$\begin{aligned} x_{i+1/2} &= (i+1/2)h, \quad i = -1,\ldots,n; \\ y_j &= jh, \quad j = 0,\ldots,n. \end{aligned}$$

For v_2, w_2^{k+1} we introduce the space H_y of admissible functions which are continuous on $\bar\Omega_2$ and bilinear on each $e_{i,j+1/2} \subset \Omega_2$. The degrees of

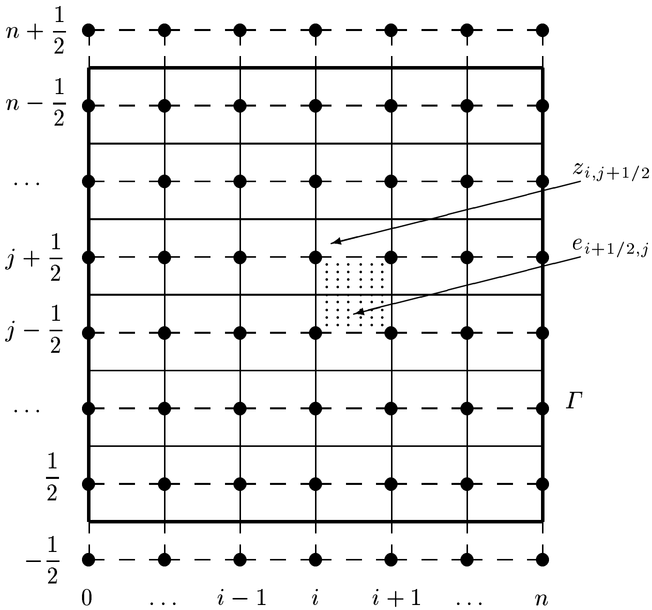

Fig. 1. Nodes $\bar{\Omega}_1^h$ of degrees of freedom for the first component of velocity (marked by sign $\bullet$).

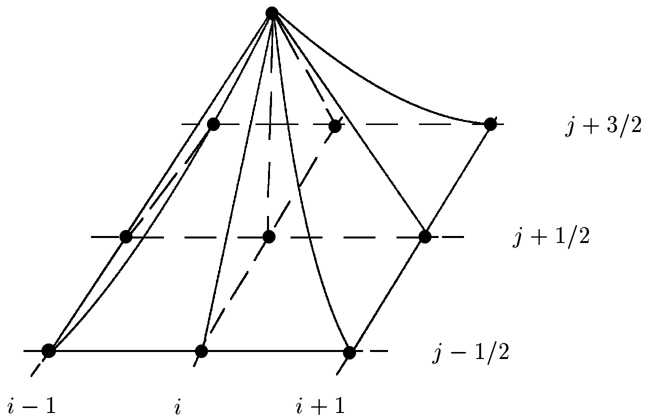

Fig. 2. Basis function $\varphi_{x,i,j+1/2}$ for the first component of velocity.

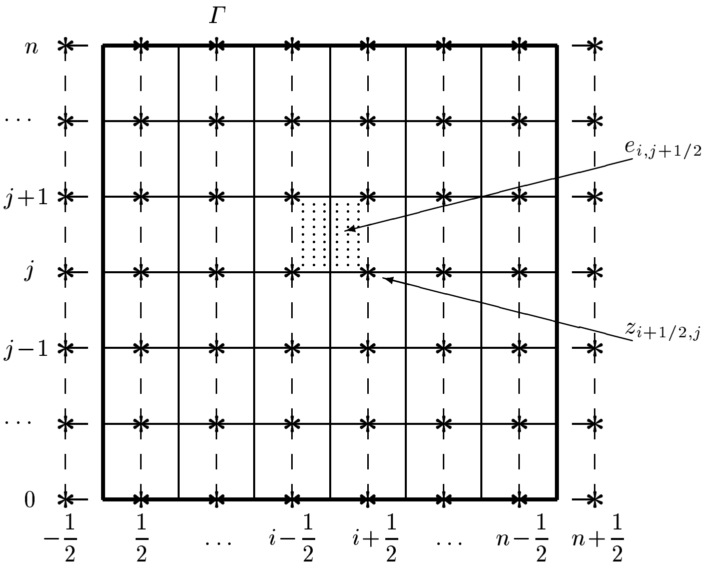

Fig. 3. Nodes $\bar{\Omega}_2^h$ of degrees of freedom for the second component of velocity (marked by sign $\ast$).

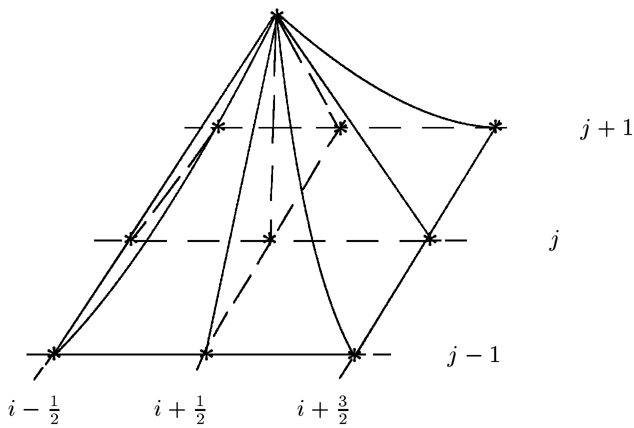

Fig. 4. Basis function $\varphi_{y,i+1/2,j}$ for the second component of velocity.

freedom of these functions are referred to the nodes $z_{i+1/2,j} = (x_{i+1/2}, y_j)$. We denote the set of these nodes

$$\bar{\Omega}_2^h = \{z_{i+1/2,j} : i = -1, \ldots, n, \ j = 0, \ldots, n\}$$

and the set of its inner nodes

$$\Omega_2^h = \bar{\Omega}_2^h \cap \Omega.$$

Then as the basis function corresponding to the node $z_{i+1/2,j}$ we take $\varphi_{y,i+1/2,j} \in H_y$ which equals 1 at $z_{i+1/2,j}$ and 0 at any other node of $\bar{\Omega}_2$.

The arrangement of nodes of $\bar{\Omega}_2$ and some basis functions from H_x are represented in Fig. 3, 4.

Finally, let us introduce the domain $\Omega_3 = (-h, 1+h) \times (-h, 1+h)$ and divide it into $(n+2)^2$ squares

$$e_{i+1/2,\,j+1/2} = (x_i, x_{i+1}) \times (y_j, y_{j+1})$$

by lines

$$x_i = ih, \ i = -1, \ldots, n+1;$$
$$y_j = jh, \ j = -1, \ldots, n+1.$$

For r, q^{k+1} we introduce the space H_p of admissible functions from $L_2(\Omega)$ which are constant on each $e_{i+1/2,j+1/2} \subset \Omega_3$. The degrees of freedom of these functions are referred to the nodes $z_{i+1/2,j+1/2} = (x_{i+1/2}, y_{j+1/2})$. We denote the set of these nodes

$$\bar{\Omega}_3^h = \{z_{i+1/2,j} : i = -1, \ldots, n, \ j = -1, \ldots, n\}$$

and introduce its inner part

$$\Omega_3^h = \bar{\Omega}_3^h \cap \Omega.$$

Then as the basis function corresponding to the node $z_{i+1/2,j+1/2}$ we take $\varphi_{p,i+1/2,j+1/2} \in H_p$ which equals 1 at $z_{i+1/2,j+1/2}$ and 0 at any other node of $\bar{\Omega}_3^h$.

The arrangement of nodes of $\bar{\Omega}_3^h$ and some basis functions from H_p are represented in Fig. 5, 6.

Introduce the grid boundary Γ^h as the set of midpoints of boundary edges

$$\Gamma^h = (\bar{\Omega}_1^h \cup \bar{\Omega}_2^h) \cap \Gamma,$$

and introduce also the scalar product for vector-functions

$$(\mathbf{u}, \mathbf{f})_\Omega = \int_\Omega (u_1 f_1 + u_2 f_2) d\Omega.$$

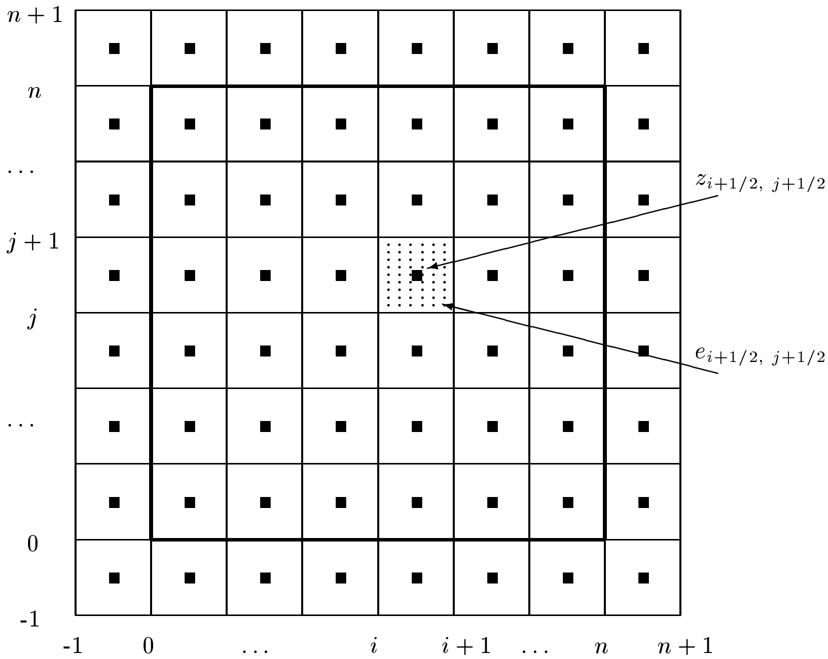

Fig. 5. Nodes $\bar{\Omega}_3^h$ of degrees of freedom for pressure (marked by sign $\blacksquare$).

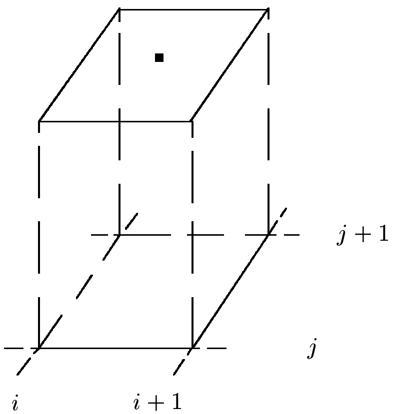

Fig. 6. Basis function $\varphi_{p,i+1/2,j+1/2}$ for pressure.

2.1 Integration over Ω

Theoretically we realize two possibilities. One of them consists in the strong integration over Ω and gives several types of discrete equations inside a domain and near a boundary. In another case the integration is implemented over a domain with a small fictitious additional subdomains that provides discrete equations to be more uniform and simpler for coding. To realize the first possibility, we formulate the Bubnov-Galerkin method for the problem (2.7) using the introduced designation: *find* $q^h(x,y) \in H_p$ *and* $\mathbf{w}^h(x,y) = (w_1^h(x,y), w_2^h(x,y))$, $w_1^h \in H_x$, $w_2^h \in H_y$, *which satisfy the boundary condition*

$$\mathbf{w}^h \cdot \mathbf{n} = \mathbf{g}^{k+1} \cdot \mathbf{n} \quad on \quad \Gamma^h \tag{2.8}$$

and the integral relation

$$\frac{1}{\tau}(\mathbf{w}^h, \mathbf{v})_\Omega - (q^h, \partial iv\ \mathbf{v})_\Omega + (\partial iv\ \mathbf{w}^h, r)_\Omega = \frac{1}{2}(\mathbf{f}^{k+1}, \mathbf{v})_\Omega + \frac{1}{\tau}(\mathbf{w}^k, \mathbf{v})_\Omega \tag{2.9}$$

for an arbitrary function $r(x,y) \in H_p$ *and for a vector-function* $\mathbf{v}(x,y) = (v_1(x,y), v_2(x,y)), v_1 \in H_x, v_2 \in H_y$, *which satisfies the boundary condition*

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad on \quad \Gamma^h. \tag{2.10}$$

Let us write the unknown functions in the form

$$\begin{aligned}
w_1^h(x,y) &= \sum_{i=0}^{n}\sum_{j=0}^{n-1} w_{1,i,j+1/2}^h \varphi_{x,i,j+1/2}(x,y), \\
w_2^h(x,y) &= \sum_{i=0}^{n-1}\sum_{j=0}^{n} w_{2,i+1/2,j}^h \varphi_{y,i+1/2,j}(x,y), \\
q^h(x,y) &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} q_{i+1/2,j+1/2}^h \varphi_{p,i+1/2,j+1/2}(x,y).
\end{aligned} \tag{2.11}$$

Then the problem (2.8) – (2.10) becomes equivalent to the system of linear algebraic equations. To get the diagonal mass matrix we shall systematically use the following quadrature formula which is the Cartesian product of the trapezium formula:

$$\int\limits_{x-h/2}^{x+h/2}\int\limits_{y-h/2}^{y+h/2} u(x,y)d\Omega \approx \frac{h^2}{4}\sum_{\pm,\pm} u(x \pm h/2,\ y \pm h/2). \tag{2.12}$$

Here the sign $\sum$ with the pointer $\pm,\pm$ means the summation of an expression with 4 possible arguments obtained by fixing of one sign + or – at each position $\pm$.

For example, let us first assemble the impact containing v_1 only, not v_2 or r. Consider the element $e_{i+1/2,j}$ and examine the first term in (2.9). Due to (2.12) we have

$$\frac{1}{\tau}\int\limits_{e_{i+1/2,j}} w_1^h v_1 d\Omega \approx \frac{h^2}{4\tau}\sum_{\pm,\pm}(w_1^h v_1)_{i+1/2\pm1/2,j\pm1/2}. \tag{2.13}$$

This gives the following impact to the left-hand side of the algebraic bilinear form:

$$[w^h_{1,i,j-1/2},\ w^h_{1,i,j+1/2},\ w^h_{1,i+1,j-1/2},\ w^h_{1,i+1,j+1/2}]$$

$$\cdot\begin{bmatrix} h^2/4\tau & 0 & 0 & 0 \\ 0 & h^2/4\tau & 0 & 0 \\ 0 & 0 & h^2/4\tau & 0 \\ 0 & 0 & 0 & h^2/4\tau \end{bmatrix}\begin{bmatrix} v_{1,i,j-1/2} \\ v_{1,i,j+1/2} \\ v_{1,i+1,j-1/2} \\ v_{1,i+1,j+1/2} \end{bmatrix}. \tag{2.14}$$

From the second term in (2.9) we get

$$\int\limits_{e_{i+1/2,j}} q^h \frac{\partial v_1}{\partial x} d\Omega \approx \sum_{\pm}\frac{h}{2}((v_{1,i+1}-v_{1,i})q^h_{i+1/2})_{j\pm1/2}. \tag{2.15}$$

This gives the following impact to the left-hand side of the algebraic bilinear form:

$$[q^h_{i+1/2,j-1/2}, q^h_{i+1/2,j+1/2}]\begin{bmatrix} -h/2 & 0 & h/2 & 0 \\ 0 & -h/2 & 0 & h/2 \end{bmatrix}$$
$$\cdot[v_{1,i,j-1/2},\ v_{1,i,j+1/2},\ v_{1,i+1,j-1/2},\ v_{1,i+1,j+1/2}]^T. \tag{2.16}$$

The sign $[\ ]^T$ means the transposition of a vector or a matrix.

Using the quadrature formula (2.12) for the right-hand side of (2.9), we get

$$\int\limits_{e_{i+1/2,j}}\left(\frac{1}{2}f_1^{k+1}v_1+\frac{1}{\tau}w_1^k v_1\right)d\Omega \approx \sum_{\pm,\pm}\left(\frac{h^2}{8}f_1^{k+1}v_1+\frac{h^2}{4\tau}w_1^k v_1\right)_{i+1/2\pm1/2,\ j\pm1/2}. \tag{2.17}$$

This gives the following impact to the right-hand side of the algebraic equality:

$$\Big[\frac{h^2}{8}f^{k+1}_{1,i,j-1/2}+\frac{h^2}{4\tau}w^k_{1,i,j-1/2},\ \frac{h^2}{8}f^{k+1}_{1,i,j+1/2}+\frac{h^2}{4\tau}w^k_{1,i,j+1/2},$$
$$\frac{h^2}{8}f^{k+1}_{1,i+1,j-1/2}+\frac{h^2}{4\tau}w^k_{1,i+1,j-1/2},\ \frac{h^2}{8}f^{k+1}_{1,i+1,\ j+1/2}+\frac{h^2}{4\tau}w^k_{1,i+1,\ j+1/2}\Big]$$
$$\cdot[v_{1,i,j-1/2},\ v_{1,i,j+1/2},\ v_{1,i+1,\ j-1/2},\ v_{1,i+1,\ j+1/2}]^T. \tag{2.18}$$

For the further study of grid equations we make its nodal assemblage. For example, in order for an arbitrary value of $v_{1,i,j+1/2}$ to satisfy the equality (2.9), one has to equate its coefficients in the right-hand and left-hand sides. For a node $z_{i,j+1/2}\in\Omega^h_1$ four elements $e_{i\pm1/2,\ j+1/2\pm1/2}$ have nonzero coefficients. Summing these coefficients over four elements, equate them:

$$\begin{aligned}&\frac{h^2}{\tau}w^h_{1,i,j+1/2}+h(q^h_{i+1/2,j+1/2}-q^h_{i-1/2,j+1/2})\\&=\frac{h^2}{\tau}w^k_{1,i,j+1/2}+\frac{h^2}{2}f^{k+1}_{1,i,j+1/2},\end{aligned}\tag{2.19}$$
$$i=1,\dots,n-1,\ j=1,\dots,n-2.$$

Now consider the impacts of elements which are cut by the boundary Γ_y, for example, $e_{i+1/2,0}$. Take into consideration twice as small domain $e_{i+1/2,0}\cap\Omega$ and transform (2.13) into the approximate equality

$$\frac{1}{\tau}\int\limits_{e_{i+1/2,0}\cap\Omega} w^h_1 v_1\,d\Omega\approx\frac{h^2}{8\tau}\sum_{\pm,\pm}(w^h_1v_1)_{i+1/2\pm1/2,\,1/4\pm1/4}. \tag{2.20}$$

This gives the following impact:

$$[w^h_{1,i,0},\ w^h_{1,i,1/2},\ w^h_{1,i+1,0},\ w^h_{1,i+1,1/2}]$$
$$\cdot\begin{bmatrix}h^2/8\tau&0&0&0\\0&h^2/8\tau&0&0\\0&0&h^2/8\tau&0\\0&0&0&h^2/8\tau\end{bmatrix}\begin{bmatrix}v_{1,i,0}\\v_{1,i,1/2}\\v_{1,i+1,0}\\v_{1,i+1,1/2}\end{bmatrix}. \tag{2.21}$$

For the second term in (2.9) we get

$$\int\limits_{e_{i+1/2,0}\cap\Omega} q^h\frac{\partial v_1}{\partial x}d\Omega\approx\frac{h}{4}q^h_{i+1/2,1/2}\sum_{\pm}(v_{1,i+1}-v_{1,i})_{1/4\pm1/4}. \tag{2.22}$$

This involves the impact

$$[q^h_{i+1/2,1/2}][-h/4,\ -h/4,\ h/4,\ h/4] \cdot [v_{1,i,0},\ v_{1,i,1/2},\ v_{1,i+1,0},\ v_{1,i+1,1/2}]^T. \tag{2.23}$$

For the right-hand side of (2.9) we have

$$\int\limits_{e_{i+1/2,0}\cap\Omega} \left(\frac{1}{2} f_1^{k+1} v_1 + \frac{1}{\tau} w_1^k v_1\right) d\Omega \approx \sum_{\pm,\pm} \left(\frac{h^2}{16} f_1^{k+1} v_1 + \frac{h^2}{8\tau} w_1^k v_1\right)_{i+\frac{1}{2}\pm\frac{1}{2},\frac{1}{4}\pm\frac{1}{4}}. \tag{2.24}$$

This gives the following impact to the right-hand side of the algebraic equality:

$$\Bigg[\frac{h^2}{16} f^{k+1}_{1,i,0} + \frac{h^2}{8\tau} w^k_{1,i,0},\ \frac{h^2}{16} f^{k+1}_{1,i,1/2} + \frac{h^2}{8\tau} w^k_{1,i,1/2}\ , \\ \frac{h^2}{16} f^{k+1}_{1,i+1,0} + \frac{h^2}{8\tau} w^k_{1,i+1,0},\ \frac{h^2}{16} f^{k+1}_{1,i+1,1/2} + \frac{h^2}{8\tau} w^k_{1,i+1,1/2}\Bigg] \\ \cdot [v_{1,i,0},\ v_{1,i,1/2},\ v_{1,i+1,0},\ v_{1,i+1,1/2}]^T. \tag{2.25}$$

Now assemble the algebraic equations corresponding to $v_{1,i,1/2},\ i = 1, \dots, n-1$, over 4 elements $e_{i\pm1/2,0} \cap \Omega$ and $e_{i\pm1/2,1}$:

$$\frac{3h^2}{4\tau} w^h_{1,i,1/2} + \frac{3h}{4}(q^h_{i+1/2,1/2} - q^h_{i-1/2,1/2}) \\ = \frac{3h^2}{4\tau} w^k_{1,i,1/2} + \frac{3h^2}{8} f^{k+1}_{1,i,1/2}, \\ i = 1, \dots, n-1. \tag{2.26}$$

The similar algebraic equations correspond to $v_{1,i,0}$ and are assembled over 2 elements $e_{i\pm1/2,0} \cap \Omega$ only:

$$\frac{h^2}{4\tau} w^h_{1,i,0} + \frac{h}{4}(q^h_{i+1/2,1/2} - q^h_{i-1/2,1/2}) \\ = \frac{h^2}{4\tau} w^k_{1,i,0} + \frac{h^2}{8} f^{k+1}_{1,i,0}, \\ i = 1, \dots, n-1. \tag{2.27}$$

The similar equations are valid near the upper part of the boundary Γ_y:

$$\begin{aligned}
&\frac{3h^2}{4\tau} w^h_{1,i,n-1/2} + \frac{3h}{4}(q^h_{i+1/2,n-1/2} - q^h_{i-1/2,n-1/2}) \\
&= \frac{3h^2}{4\tau} w^k_{1,i,n-1/2} + \frac{3h^2}{8} f^{k+1}_{1,i,n-1/2}, \qquad (2.28) \\
&\frac{h^2}{4\tau} w^h_{1,i,n} + \frac{h}{4}(q^h_{i+1/2,n-1/2} - q^h_{i-1/2,n-1/2}) \\
&= \frac{h^2}{4\tau} w^k_{1,i,n} + \frac{h^2}{8} f^{k+1}_{1,i,n}, \qquad (2.29) \\
&i = 1, \ldots, n-1.
\end{aligned}$$

Analogously, the equation corresponding to $v_{2,i+1/2,j}$ inside of the domain Ω is formulated in terms of the coefficients of the stiffness matrix and the right-hand side of four elements $e_{i+1/2\pm1/2,\ j\pm1/2}$:

$$\begin{aligned}
&\frac{h^2}{\tau} w^h_{2,i+1/2,j} + h(q^h_{i+1/2,j+1/2} - q^h_{i+1/2,j-1/2}) \\
&= \frac{h^2}{\tau} w^k_{2,i+1/2,j} + \frac{h^2}{2} f^{k+1}_{2,i+1/2,j}, \qquad (2.30) \\
&i = 1, \ldots, n-2,\ j = 1, \ldots, n-1.
\end{aligned}$$

Near the boundary Γ_y we get four types of equations which are similar to (2.26)–(2.29):

$$\begin{aligned}
&\frac{3h^2}{4\tau} w^h_{2,1/2,j} + \frac{3h}{4}(q^h_{1/2,j+1/2} - q^h_{1/2,j-1/2}) \\
&= \frac{3h^2}{4\tau} w^k_{2,1/2,j} + \frac{3h^2}{8} f^{k+1}_{2,1/2,j}, \qquad (2.31) \\
&\frac{h^2}{4\tau} w^h_{2,0,j} + \frac{h}{4}(q^h_{1/2,j+1/2} - q^h_{1/2,j-1/2}) \\
&= \frac{h^2}{4\tau} w^k_{2,0,j} + \frac{h^2}{8} f^{k+1}_{2,0,j}, \qquad (2.32)
\end{aligned}$$

$$\frac{3h^2}{4\tau} w^h_{2,n-1/2,j} + \frac{3h}{4}(q^h_{n-1/2,j+1/2} - q^h_{n-1/2,j-1/2})$$
$$= \frac{3h^2}{4\tau} w^k_{2,n-1/2,j} + \frac{3h^2}{8} f^{k+1}_{2,n-1/2,j}, \tag{2.33}$$
$$\frac{h^2}{4\tau} w^h_{2,n,j} + \frac{h}{4}(q^h_{n-1/2,j+1/2} - q^h_{n-1/2,j-1/2})$$
$$= \frac{h^2}{4\tau} w^k_{2,n,j} + \frac{h^2}{8} f^{k+1}_{2,n,j}, \tag{2.34}$$
$$i = 1, \ldots, n-1.$$

Now let us study the impact containing r. This time we use the following quadrature formula which is the Cartesian product of the central rectangle formula:

$$\int\limits_{x-h/2}^{x+h/2} \int\limits_{y-h/2}^{y+h/2} u(x,y) d\Omega \approx h^2 u(x,y). \tag{2.35}$$

Consider the element $e_{i+1/2,\ j+1/2}$ and examine the third term in (2.9). Due to (2.35) we have

$$\int\limits_{e_{i+1/2,\ j+1/2}} r\ div\ w^h d\Omega$$
$$\approx r_{i+1/2,\ j+1/2}(w^h_{1,i+1,\ j+1/2} - w^h_{1,i,j+1/2} + w^h_{2,i+1/2,\ j+1} - w^h_{2,i+1/2,j}). \tag{2.36}$$

This gives the following impact to the left-hand side of the algebraic bilinear form:

$$[w^h_{1,i,j+1/2},\ w^h_{1,i+1,j+1/2},\ w^h_{2,i+1/2,j},\ w^h_{2,i+1/2,\ j+1}]$$
$$\cdot[-h,\ h,\ -h,\ h]^T \cdot [r_{i+1/2,\ j+1/2}]. \tag{2.37}$$

This formula contains all entries of the stiffness matrix associated with $r_{i+1/2,\ j+1/2}$. Therefore the assembly of the nodal equations corresponding to $r_{i+1/2,\ j+1/2}$ gives

$$h(w^h_{1,i+1,j+1/2} - w^h_{1,i,j+1/2}) + h(w^h_{2,i+1/2,j+1} - w^h_{2,i+1/2,j}) = 0. \tag{2.38}$$

Now we consider the boundary condition (2.8). First we introduce the discrete analogue of Γ_x, Γ_y:

$$\Gamma^h_x = \bar{\Omega}^h_1 \cap \Gamma \cup \{z_{0,0},\ z_{0,n},\ z_{n,0},\ z_{n,n}\},$$
$$\Gamma^h_y = \bar{\Omega}^h_2 \cap \Gamma \cup \{z_{0,0},\ z_{0,n},\ z_{n,0},\ z_{n,n}\}.$$

Doing the simplifications which are connected with the concrete form of normal vector, we get

$$w_1^h = g_1^{k+1} \quad \text{on} \quad \Gamma_x^h, \tag{2.39}$$
$$w_2^h = g_2^{k+1} \quad \text{on} \quad \Gamma_y^h. \tag{2.40}$$

The question of consequence of the boundary condition (2.10) arises. For example, consider the nearboundary cell $e_{n-1/2,j}$. From the concrete form of the external normal (1, 0) and the condition (2.10) at the node $(x_n,\ y_{j+1/2})$ it follows that

$$v_{1,n,j+1/2} = 0. \tag{2.41}$$

Hence, for any coefficients the terms containing $v_{1,n,j+1/2}$ in the both sides of the equality (2.9) do not give an equation corresponding to this value (or what is the same to the node $z_{n,j+1/2}$). Analogously, for the near-boundary cell $e_{i,n-1/2}$ we have

$$v_{2,i+1/2,n} = 0. \tag{2.42}$$

Here *this* value turns to zero and there is no grid equation corresponding to it for $\mathbf{w}^h, q^h$. At last, both situations (2.41), (2.42) take place at the same time for the node $e_{n-1/2,n-1/2}$ and no equation exists for two nodes $z_{n,n-1/2}$ and $z_{n-1/2,n}$. One of three situations takes place along all grid boundary Γ^h.

To do the grid equations (2.19) – (2.38) more habitual we introduce the following notations:

$$u_{\overset{\circ}{x}}(x) = (u(x+h/2) - u(x-h/2))/h,$$
$$u_{\overset{\circ}{y}}(y) = (u(y+h/2) - u(y-h/2))/h,$$

$$\begin{aligned}
&\bar{w}^h_{1,i,1/2} = \frac{3}{4} w^h_{1,i,1/2} + \frac{1}{4} w^h_{1,i,0};\\
&\bar{w}^h_{1,i,n-1/2} = \frac{3}{4} w^h_{1,i,n-1/2} + \frac{1}{4} w^h_{1,i,n};\\
&\bar{w}^h_{1,i,j+1/2} = w^h_{1,i,j+1/2},\ j = 0, \ldots, n-1;\\
&i = 0, \ldots, n;
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
&\bar{w}^h_{2,1/2,j} = \frac{3}{4} w^h_{2,1/2,j} + \frac{1}{4} w^h_{2,0,j};\\
&\bar{w}^h_{2,n-1/2,j} = \frac{3}{4} w^h_{2,n-1/2,j} + \frac{1}{4} w^h_{2,n,j};\\
&\bar{w}^h_{2,i+1/2,j} = w^h_{2,i+1/2,j},\ i = 0, \ldots, n-1;\\
&j = 0, \ldots, n;
\end{aligned} \tag{2.44}$$

and similar formulae for $\bar{f}_1^{k+1}$, $\bar{f}_2^{k+1}$, $\bar{w}_1^k$, $\bar{w}_2^k$, $\bar{g}_1^{k+1}$, $\bar{g}_2^{k+1}$. Combining the equations (2.26) and (2.27) we get the equation like (2.19):

$$
\begin{aligned}
&\frac{h^2}{\tau}\bar{w}^h_{1,i,1/2} + h(q^h_{i+1/2,1/2} - q^h_{i-1/2,1/2}) \\
&= \frac{h^2}{\tau}\bar{w}^k_{1,i,1/2} + \frac{h^2}{2}\bar{f}^{k+1}_{1,i,1/2}, \\
&i = 1, \dots, n-1.
\end{aligned} \tag{2.45}
$$

The similar equations are obtained by combining of (2.28) and (2.29), (2.31) and (2.32), (2.33) and (2.34). Dividing the obtained equations by h^2 we get

$$\frac{1}{\tau}\bar{w}_1^h + q^h_{\overset{\circ}{x}} = \frac{1}{\tau}\bar{w}_1^k + \frac{1}{2}\bar{f}_1^{k+1} \quad \text{on} \quad \Omega_1^h, \tag{2.46}$$

$$\frac{1}{\tau}\bar{w}_2^h + q^h_{\overset{\circ}{y}} = \frac{1}{\tau}\bar{w}_2^k + \frac{1}{2}\bar{f}_2^{k+1} \quad \text{on} \quad \Omega_2^h, \tag{2.47}$$

$$(\bar{w}_1^h)_{\overset{\circ}{x}} + (\bar{w}_2^h)_{\overset{\circ}{y}} = 0 \quad \text{on} \quad \Omega_3^h. \tag{2.48}$$

The boundary conditions follow from (2.39) – (2.40):

$$\bar{w}_1^h = \bar{g}_1^{k+1} \quad \text{on} \quad \Gamma_x^h, \tag{2.49}$$
$$\bar{w}_2^h = \bar{g}_2^{k+1} \quad \text{on} \quad \Gamma_y^h. \tag{2.50}$$

It should be noted that we obtained the difference scheme with staggerred nodes which was very popular at the end of 1970-s and at the beginning of 1980-s.

Enumerate the nodes Ω_1^h, Ω_2^h, Ω_3^h in lexicographical order from 1 to $3n^2 - 2n$. In (2.48) rearrange the terms which are known due to (2.49), (2.50), to the right-hand side. As a result, the grid problem (2.41)–(2.50) can be written in the following matrix-vector form:

$$\begin{bmatrix} \frac{1}{\tau}E_{n(n-1)} & \cdot & A_1 \\ \cdot & \frac{1}{\tau}E_{n(n-1)} & A_2 \\ -A_1^T & -A_2^T & \cdot \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ Q \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}. \tag{2.51}$$

Here

$$W_1 = (\bar{w}_1^h(z_{1,1/2}), \bar{w}_1^h(z_{1,3/2}), \dots, \bar{w}_1^h(z_{n-1,n-1/2}))^T;$$

$$W_2 = (\bar{w}_2^h(z_{1/2,1}), \bar{w}_2^h(z_{1/2,2}), \dots, \bar{w}_2^h(z_{n-1/2,n-1}))^T;$$

$$Q = (q^h(z_{1/2,1/2}), q^h(z_{1/2,3/2}), \dots, q^h(z_{n-1/2,n-1/2}))^T;$$

$$F_1 = (F_1(z_{1,1/2}), F_1(z_{1,3/2}), \dots, F_1(z_{n-1,n-1/2}))^T,$$

where $F_1(z_{i,j+1/2}) = \dfrac{1}{\tau}\bar{w}_1^k(z_{i,j+1/2}) + \dfrac{1}{2}\bar{f}_1^{k+1}(z_{i,j+1/2});$

$$F_2 = (F_2(z_{1/2,1}), F_2(z_{1/2,2}), \dots, F_2(z_{n-1/2,n-1}))^T,$$

where $F_2(z_{i+1/2,j}) = \dfrac{1}{\tau}\bar{w}_2^k(z_{i+1/2,j}) + \dfrac{1}{2}\bar{f}_2^{k+1}(z_{i+1/2,j});$

$$F_3 = (F_3(z_{1/2,1/2}), F_3(z_{1/2,3/2}), \dots, F_3(z_{n-1/2,n-1/2}))^T,$$

where $F_3(z_{i+1/2,j+1/2}) = \delta_{i,0}\dfrac{1}{h}\bar{g}_1^{k+1}(z_{0,j+1/2}) - \delta_{i,n-1}\dfrac{1}{h}\bar{g}_1^{k+1}(z_{n,j+1/2})$

$$+\delta_{0,j}\frac{1}{h}\bar{g}_2^{k+1}(z_{i+1/2,0}) - \delta_{n-1,j}\frac{1}{h}\bar{g}_2^{k+1}(z_{i+1/2,n});$$

here δ_{ij} is the Kronecker symbol that equals 1 for $i = j$ and 0 otherwise; $E_{n(n-1)}$ is the $n(n-1)\times n(n-1)$ identity matrix; A_1 is an $n(n-1)\times n(n-1)$ matrix of the block form

$$A_1 = \frac{1}{h}E_n \otimes B = \frac{1}{h}\begin{bmatrix} \circ & E_n & & & \\ -E_n & \circ & E_n & & \\ & \ddots & \ddots & \ddots & \\ & & -E_n & \circ & E_n \\ & & & -E_n & \circ \end{bmatrix},$$

A_2 is an $n(n-1)\times(n-1)$ matrix of the block-diagonal form:

$$A_2 = \frac{1}{h}B \otimes E_n = \frac{1}{h}\begin{bmatrix} B & & \\ & \ddots & \\ & & B \end{bmatrix},$$

where $\otimes$ is the tensor product; B is an $(n-1)\times(n-1)$ matrix of the form:

$$B = \begin{bmatrix} \cdot & 1 & & & & \\ -1 & \cdot & 1 & & & \\ & \ddots & \cdot & \ddots & & \\ & & \ddots & \cdot & \ddots & \\ & & & -1 & \cdot & 1 \\ & & & & -1 & \cdot \end{bmatrix}.$$

Let us prove that the problems (2.51) and (2.41)–(2.50) are stable with respect to the initial data and the right-hand side $\mathbf{f}^{k+1}$ for the components of a speed vector. For this purpose we introduce grid norms which are analogous to functional L_2-norms:

$$\|\mathbf{w}\|_{L_2,h} = (\|w_1\|_{1,h}^2 + \|w_2\|_{2,h}^2)^{1/2}$$

where

$$\|w_1\|_{1,h}^2 = h^2 \sum_{i=1}^{n-1} \sum_{j=0}^{n-1} w_1^2(z_{i,j+1/2}), \tag{2.52}$$

$$\|w_2\|_{2,h}^2 = h^2 \sum_{i=0}^{n-1} \sum_{j=1}^{n-1} w_2^2(z_{i+1/2,j}), \tag{2.53}$$

and

$$\|q\|_{3,h}^2 = h^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} p^2(z_{i+1/2,j+1/2}). \tag{2.54}$$

Theorem 1. *If*

$$g_1^{k+1} = 0 \quad on \quad \Gamma_x^h, \quad g_2^{k+1} = 0 \quad on \quad \Gamma_y^h \tag{2.55}$$

for the problem (2.41)–(2.50) *then the following a priori estimate holds:*

$$\|\bar{\mathbf{w}}^h\|_{L_2,h} \le \|\bar{\mathbf{w}}^k\|_{L_2,h} + \frac{\tau}{2}\|\bar{\mathbf{f}}^{k+1}\|_{L_2,h}. \tag{2.56}$$

Proof. Introduce the scalar products:

$$(u,v)_{1,h} = \sum_{i=1}^{n-1} \sum_{j=0}^{n-1} u(z_{i,j+1/2})v(z_{i,j+1/2}),$$

$$(u,v)_{2,h} = \sum_{i=0}^{n-1} \sum_{j=1}^{n-1} u(z_{i+1/2,j})v(z_{i+1/2,j}),$$

$$(p,q)_{3,h} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} p(z_{i+1/2,j+1/2})q(z_{i+1/2,j+1/2}).$$

Owing to the zero boundary value (2.55), the simple regrouping of terms yields two equalities which are the grid analogues of the integration by parts formula:

$$
\begin{aligned}
((w_1)_{\overset{\circ}{x}}, p)_{3,h} &= -(w_1, p_{\overset{\circ}{x}})_{1,h}, \\
((w_2)_{\overset{\circ}{y}}, p)_{3,h} &= -(w_2, p_{\overset{\circ}{y}})_{2,h}.
\end{aligned}
\tag{2.57}
$$

Multiply the equation (2.46) by w_1^h and sum up over Ω_1^h:

$$
\frac{1}{\tau}(\bar{w}_1^h, \bar{w}_1^h)_{1,h} + (q^h_{\overset{\circ}{x}}, \bar{w}_1^h)_{1,h} = \frac{1}{\tau}(\bar{w}_1^k, \bar{w}_1^h)_{1,h} + \frac{1}{2}(\bar{f}_1^{k+1}, \bar{w}_1^h)_{1,h}. \tag{2.58}
$$

The equation (2.47) is mulpiplied by w_2^h and is summed up over Ω_2^h:

$$
\frac{1}{\tau}(\bar{w}_2^h, \bar{w}_2^h)_{2,h} + (q^h_{\overset{\circ}{y}}, \bar{w}_2^h)_{2,h} = \frac{1}{\tau}(\bar{w}_2^k, \bar{w}_2^h)_{2,h} + \frac{1}{2}(\bar{f}_2^{k+1}, \bar{w}_2^h)_{2,h}. \tag{2.59}
$$

The equation (2.48) is multiplied by q^h and is summed up over Ω_3^h:

$$
((\bar{w}_1^h)_{\overset{\circ}{x}}, q^h)_{3,h} + ((\bar{w}_2^h)_{\overset{\circ}{y}}, q^h)_{3,h} = 0. \tag{2.60}
$$

Now we combine the equalities (2.58) – (2.60) and apply the grid formulae of integration by parts. As a result,

$$
\begin{aligned}
&\frac{1}{\tau}(\bar{w}_1^h, \bar{w}_1^h)_{1,h} + \frac{1}{\tau}(\bar{w}_2^h, \bar{w}_2^h)_{2,h} = \frac{1}{\tau}(\bar{w}_1^k, \bar{w}_1^h)_{1,h} \\
&+\frac{1}{\tau}(\bar{w}_2^k, \bar{w}_2^h)_{2,h} + \frac{1}{2}(\bar{f}_1^{k+1}, \bar{w}_1^h)_{1,h} + \frac{1}{2}(\bar{f}_2^{k+1}, \bar{w}_2^h)_{2,h}.
\end{aligned}
\tag{2.61}
$$

We multiply this equality by τ and apply the Cauchy-Bunjakowski inequality to every term of the right-hand side:

$$
\begin{aligned}
&\|\bar{w}_1^h\|_{1,h}^2 + \|\bar{w}_2^h\|_{2,h}^2 \le \|\bar{w}_1^k\|_{1,h}\|\bar{w}_1^h\|_{1,h} + \|\bar{w}_2^k\|_{2,h}\|\bar{w}_2^h\|_{2,h} \\
&+\frac{\tau}{2}\|\bar{f}_1^{k+1}\|_{1,h}\|\bar{w}_1^h\|_{1,h} + \frac{\tau}{2}\|\bar{f}_2^{k+1}\|_{2,h}\|\bar{w}_2^h\|_{2,h}.
\end{aligned}
\tag{2.62}
$$

We again apply the Cauchy-Bunjakowski inequality to two pairs of terms of the right-hand side and cancel the coincident multiplier:

$$
\begin{aligned}
&(\|\bar{w}_1^h\|_{1,h}^2 + \|\bar{w}_2^h\|_{2,h}^2)^{1/2} \le (\|\bar{w}_1^k\|_{1,h}^2 + \|\bar{w}_2^k\|_{2,h}^2)^{1/2} \\
&+\frac{\tau}{2}(\|\bar{f}_1^{k+1}\|_{1,h}^2 + \|\bar{f}_2^{k+1}\|_{2,h}^2)^{1/2}.
\end{aligned}
$$

In view of the accepted notations this inequality coincides with (2.56). □

Now we construct the problem for determination of pressure and consider the question of its stability. To do this, take the difference derivative $(\cdot)_{\overset{\circ}{x}}$ of (2.46) at nodes of Ω_3^h:

$$(\bar{w}_1^h)_{\overset{\circ}{x}} + \tau (q^h_{\overset{\circ}{x}})_{\overset{\circ}{x}} = (\bar{w}_1^k)_{\overset{\circ}{x}} + \frac{\tau}{2}(\bar{f}_1^{k+1})_{\overset{\circ}{x}}. \tag{2.63}$$

To define the derivative $(\bar{w}_2^h)_{\overset{\circ}{y}}$ we take the difference derivative $(\cdot)_{\overset{\circ}{y}}$ of (2.47):

$$(\bar{w}_2^h)_{\overset{\circ}{y}} + \tau (q^h_{\overset{\circ}{y}})_{\overset{\circ}{y}} = (\bar{w}_2^k)_{\overset{\circ}{y}} + \frac{\tau}{2}(\bar{f}_2^{k+1})_{\overset{\circ}{y}}. \tag{2.64}$$

Now we eliminate $(\bar{w}_1^h)_{\overset{\circ}{x}}$ and $(\bar{w}_2^h)_{\overset{\circ}{y}}$ in (2.48), divide the obtained equality by τ, and rearrange the known expressions to the right-hand side. As a result we get

$$\begin{aligned} -(q^h_{\overset{\circ}{x}})_{\overset{\circ}{x}} - (q^h_{\overset{\circ}{y}})_{\overset{\circ}{y}} &= -\frac{1}{\tau}(\bar{w}_1^k)_{\overset{\circ}{x}} - \frac{1}{\tau}(\bar{w}_2^k)_{\overset{\circ}{y}} \\ & -\frac{1}{2}(\bar{f}_1^{k+1})_{\overset{\circ}{x}} - \frac{1}{2}(\bar{f}_2^{k+1})_{\overset{\circ}{y}} \quad \text{on} \quad \Omega_3^h. \end{aligned} \tag{2.65}$$

And at the nodes of Γ_x^h and Γ_y^h the other conditions of Neumann type follow from (2.46), (2.49) and (2.47), (2.50). For example, on Γ_x^h from (2.46) and (2.49) it follows that

$$q^h_{\overset{\circ}{x}} = -\frac{1}{\tau}\bar{g}_1^{k+1} + \frac{1}{\tau}\bar{w}_1^k + \frac{1}{2}\bar{f}_1^{k+1} \quad \text{on} \quad \Gamma_x^h. \tag{2.66}$$

On Γ_y^h from (2.47) and (2.50) it follows that

$$q^h_{\overset{\circ}{y}} = -\frac{1}{\tau}\bar{g}_2^{k+1} + \frac{1}{\tau}\bar{w}_2^k + \frac{1}{2}\bar{f}_2^{k+1} \quad \text{on} \quad \Gamma_y^h. \tag{2.67}$$

The system of linear algebraic equations (2.65) – (2.67) can be reduced to the (Schur complement) system

$$BQ = G \tag{2.68}$$

with the symmetric matrix

$$B = -A_1^T A_1 - A_2^T A_2 \tag{2.69}$$

and the right-hand side

$$G = \frac{1}{\tau}F_3 + A_1^T F_1 + A_2^T F_2. \tag{2.70}$$

This matrix is the same one as for the discrete Poisson equation with the Neumann boundary condition. It is well-known that this matrix is singular, the dimension of its kernel equals 1, and the basis of the kernal consists of only constant n^2-vector

$$S = (1, \dots, 1). \tag{2.71}$$

Thus, the system (2.68) has a solution if and only if the right-hand side G is orthogonal to S. It means that

$$\sum_{i,j=1}^{n} G_{ij} = 0. \tag{2.72}$$

Let this be valid. Then the system (2.68) has the infinite number of solutions. We take (normal) one which is orthogonal to S:

$$\sum_{i,j=1}^{n} Q_{ij} = 0. \tag{2.73}$$

Note that this equality is the discrete analogue of the condition (1.5).

In Theorem 1 we considered the impact of initial values and the right-hand side $\mathbf{f}$ when computing $\mathbf{u}$. Now let us study the situation when a non-zero right-hand side arises in (2.43) owing to an approximation (truncation) error or to a residual of iterative process. For this purpose consider the problem

$$\frac{1}{\tau} z_1^h + r^h_{\overset{\circ}{x}} = 0 \quad \text{on} \quad \Omega_1^h, \tag{2.74}$$

$$\frac{1}{\tau} z_2^h + r^h_{\overset{\circ}{y}} = 0 \quad \text{on} \quad \Omega_2^h, \tag{2.75}$$

$$(z_1^h)_{\overset{\circ}{x}} + (z_2^h)_{\overset{\circ}{y}} = \psi^h \quad \text{on} \quad \Omega_3^h, \tag{2.76}$$

$$z_1^h = 0 \quad \text{on} \quad \Gamma_x^h, \tag{2.77}$$

$$z_2^h = 0 \quad \text{on} \quad \Gamma_y^h. \tag{2.78}$$

Here a grid function ψ^h is defined on Ω_3^h; $\mathbf{z}^h = (z_1^h, z_2^h)$.

Theorem 2. *For the problem* (2.74) – (2.78) *the following a priori estimate holds:*

$$\|\mathbf{z}^h\|_{L_{2,h}} \le c_1 \|\psi^h\|_{3,h} \tag{2.79}$$

where a constant c_1 depends on Ω only.

Proof. Let us multiply the equation (2.74) by z_1^h and sum up over Ω_1^h:

$$\frac{1}{\tau}(z_1^h, z_1^h)_{1,h} + (r^h_{\overset{\circ}{x}}, z_1^h)_{1,h} = 0. \tag{2.80}$$

The equation (2.75) is multiplied by z_2^h and is summed up over Ω_2^h:

$$\frac{1}{\tau}(z_2^h, z_2^h)_{2,h} + (r^h_{\overset{\circ}{y}}, z_2^h)_{2,h} = 0. \tag{2.81}$$

The equation (2.76) is multiplied by r^h and is summed up over Ω_3:

$$((z_1^h)_{\overset{\circ}{x}}, r^h)_{3,h} + ((z_2^h)_{\overset{\circ}{y}}, r^h)_{3,h} = (\psi^h, r^h)_{3,h}. \tag{2.82}$$

Now we combine the equalities (2.80) – (2.82) and apply the grid formulae (2.57) of integration by parts with information (2.77), (2.78):

$$\frac{1}{\tau}(z_1^h, z_1^h)_{1,h} + \frac{1}{\tau}(z_2^h, z_2^h)_{2,h} = (\psi^h, r^h)_{3,h}. \tag{2.83}$$

Multiply this equality by τ and apply the Cauchy-Bunyakovskii inequality for the right-hand side:

$$\|z_1^h\|_{1,h}^2 + \|z_2^h\|_{2,h}^2 \le \tau\|\psi^h\|_{3,h}\|r^h\|_{3,h}. \tag{2.84}$$

Then we consider the reduced (Schur complement) system

$$\begin{aligned}
&-(r^h_{\overset{\circ}{x}})_{\overset{\circ}{x}} - (r^h_{\overset{\circ}{y}})_{\overset{\circ}{y}} = \frac{1}{\tau}\psi^h \quad \text{on} \quad \Omega_3^h,\\
&r^h_{\overset{\circ}{x}} = 0 \quad \text{on} \quad \Gamma_x^h,\\
&r^h_{\overset{\circ}{y}} = 0 \quad \text{on} \quad \Gamma_y^h.
\end{aligned} \tag{2.85}$$

Let us orthogonalize ψ^h to the function $s^h = 1$ on Ω_3^h:

$$\psi_1^h = \psi^h - \frac{(\psi^h, s^h)_{3,h}}{\|s^h\|_{3,h}^2}s^h.$$

In so doing the discrete L_2-norm does not increase:

$$\|\psi_1^h\|_{3,h} \le \|\psi^h\|_{3,h}.$$

The minimal *nonzero* eigenvalue of the matrix B in (2.85) is positive and can be bounded from below by a constant c_2 independent of τ, h, and t. Therefore the normal solution of the problem (2.85) is estimated as follows:

$$\|r^h\|_{3,h} \le \frac{1}{\tau c_2}\|\psi_1^h\|_{3,h} \le \frac{1}{\tau c_2}\|\psi^h\|_{3,h}. \tag{2.86}$$

Combining this inequality with (2.84), we get

$$\|\mathbf{z}^h\|^2_{L_2,h} \leq \frac{1}{c_2}\|\psi^h\|^2_{3,h}.$$

Take the square root and put $c_1 = 1/\sqrt{c_2}$ to finish the proof. □

Solving the systems (2.65) – (2.67) we obtain the grid function of pressure q^h at nodes Ω_3^h at time level t_{k+1}. After that, by formulae (2.46) and (2.47) we calculate the grid functions w_1^h and w_2^h obviously. This calculation conclude the description of fractional step of pressure work.

Remark 2. It should be noted that the special placing of nodes ensures the stability of computation of pressure (see, for example, [31], [2]). □

Remark 3. The equations (2.65) contains the expression

$$-\frac{1}{\tau}(w_1^k)_{\overset{\circ}{x}} - \frac{1}{\tau}(w_2^k)_{\overset{\circ}{y}}, \tag{2.87}$$

which should equal 0 in consequence of the equality analogous to (2.48) at level $t = t_k$. But in the consequent estimation we shall use the iterative methods for solving algebraic systems, which do not yield the exact equality of the expression (2.87) to 0. This expression may be small, but its contribution to the approximate solution has the tendency to the linear accumulation from level to level along the time axis. Therefore, supposing that (2.87) equals zero in (2.65), we slightly decrease the number of arithmetic operations at every level, but we introduce an additional source of error, which imposes requirements on an iterative process and leads to an increase of number its iteration steps. □

Remark 4. The formulae (2.48) and (2.49) contain the difference differentiation of the function q^h. It may considerably decrease the accuracy of calculations of w_1^h and w_2^h in comparison with q^h. To reduce this effect, one should apply an iterative process which minimizes the error of iterative approximation in the grid energy norm. □

2.2 Integration with the help of small fictitious domains for uniformity of equations

To realize the approach with fictitious domains first we consider extended domain $\Omega_1 = (0,1) \times (-h/2, 1+h/2)$ and prolong the equation (2.2) by smooth way into additional strips. For this purpose we prolong w_1^k, w_1^{k+1}, $\partial q^{k+1}/\partial x$ through boundary Γ_y using Taylor expansions of these functions in direction y. After that we compute

$$f_1^{k+1} = 2\left(\frac{1}{\tau}w_1^{k+1} - \frac{1}{\tau}w_1^k + \frac{\partial q^{k+1}}{\partial x}\right)$$

in two strips $\Omega_1 \setminus \Omega$. Thus, we have equation (2.2) to be valid in extended domain Ω_1. Similarly by Taylor expansions we prolong boundary function g_1^{k+1} on 4 segments $\{0,1\} \times (-h/2, 0)$ and $\{0,1\} \times (1, 1+h/2)$. It gives boundary condition

$$w_1 = g_1^{k+1} \text{ on extended segments } \{0,1\} \times (-h/2, 1+h/2). \tag{2.88}$$

To simplify representation we put $v_2 = 0$ and $r = 0$ in integral relation like (2.9) and obtain the following Galerkin formulation: *find* $q^h(x,y) \in H_p$ *and* $w_1^h(x,y) \in H_x$ *which satisfy the boundary condition* (2.88) *and the integral relation*

$$\frac{1}{\tau}(w_1^h, v_1)_{\Omega_1} - (q^h, \partial v_1/\partial x)_{\Omega_1} = \frac{1}{2}(f_1^{k+1}, v_1)_{\Omega_1} + \frac{1}{\tau}(w_1^h, v_1)_{\Omega_1} \tag{2.89}$$

for an arbitrary function $v_1(x,y) \in H_x$ *which satisfies the boundary condition*

$$v_1 = 0 \textit{ on extended segments } \{0,1\} \times (-h/2, 1+h/2). \tag{2.90}$$

Repeating considerations (2.13)–(2.29) on the extended domain Ω_1 we get the same equations like (2.19) for $j = 0, \ldots, n-1$ and some equations for $j = -1/2$ and $j = n + 1/2$. Last equations does not influence on approximate solution in internal nodes and we shall omit them in our algorithmic constructions. Thus, this way with small fictitious domains gives the uniform equations in all internal nodes of Ω_1^h

$$\begin{aligned} &\frac{h^2}{\tau} w_{1,i,j+1/2}^h + h(q_{i+1/2,j+1/2}^h - q_{i-1/2,j+1/2}^h) \\ &= \frac{h^2}{\tau} w_{1,i,j+1/2}^k + \frac{h^2}{2} f_{1,i,j+1/2}^{k+1}, \\ &i = 1, \ldots, n-1, \; j = 0, \ldots, n-1. \end{aligned} \tag{2.91}$$

with boundary conditions

$$\begin{aligned} &w_{1,i,j+1/2}^h = g_{1,i,j+1/2}^{k+1}, \\ &i = 0, n, \; j = 0, \ldots, n-1. \end{aligned} \tag{2.92}$$

Note that in this equations we does not use any data from fictitious domains therefore we need them only from theoretical point of view without algorithmic complication.

Second, we consider extended domain $\Omega_2 = (-h/2, 1+h/2) \times (0,1)$, prolong equation (2.3) by smooth way into additional strips $\Omega_2 \setminus \Omega$, and prolong boundary function g_2^{k+1} on extended segments $(-h/2, 1+h/2) \times \{0,1\}$. Thus, we get equation (2.3) to be valid on extended domain Ω_2 and the boundary condition

$$w_2^h = g_2^{k+1} \text{ on extended segments } (-h/2, 1+h/2) \quad \times \{0,1\} \qquad (2.93)$$

Again we put $v_1 = 0$ and $r = 0$ in integral relation (2.9), take Ω_2 instead of Ω, and obtain the following Galerkin formulation: *find* $q^h(x,y) \in H_p$ *and* $w_2^h(x,y) \in H_y$ *which satisfy boundary condition* (2.93) *and the integral relation*

$$\frac{1}{\tau}(w_2^h, v_2)_{\Omega_2} - (q^h, \partial v_2/\partial y)_{\Omega_2} = \frac{1}{2}(f_2^{k+1}, v_2)_{\Omega_2} + \frac{1}{\tau}(w_2^k, v_2)_{\Omega_2} \qquad (2.94)$$

for an arbitrary function $v_2 \in H_y$ *which satisfies the boundary condition*

$$v_2 = 0 \text{ on extended segments } (-h/2, 1+h/2) \quad \times \{0,1\}. \qquad (2.95)$$

Repeating considerations like (2.13)–(2.29) on the extended domain Ω_2 we get the same equations like (2.30) for $i = 0, \ldots, n-1$ and some equations for $i = -1/2$ and $i = n+1/2$. Last equations again does not influence on approximate solution in internal nodes and we shall omit them in our further algorithmic constructions. So, we get by the approach with small fictitious domains the uniform equations in all internal nodes of Ω_2^h:

$$\begin{aligned} &\frac{h^2}{\tau} w_{2,i+1/2,j}^h + h(q_{i+1/2,j+1/2}^h - q_{i+1/2,j-1/2}^h) \\ &= \frac{h^2}{\tau} w_{2,i+1/2,j}^k + \frac{h^2}{\tau} f_{2,i+1/2,j}^{k+1}, \\ &\qquad i = 0, \ldots, n-1, \ j = 1, \ldots, n-1, \end{aligned} \qquad (2.96)$$

with boundary conditions

$$\begin{aligned} &w_{2,i+1/2,j}^h = g_{2,i+1/2,j}^{k+1}, \\ &i = 0, \ldots, n-1, \quad j = 0, n. \end{aligned} \qquad (2.97)$$

The derivation of algebraic equations from (2.4) stays the same as in (2.36)–(2.38).

One can see that we directly obtain the system of algebraic equations like (2.46)–(2.50) then all representations and conclusions (2.52)–(2.87) are valid within change $\bar{w}_i$ by w_i. Thus, this problem is stable due to Theorem 1, 2.

So, we indeed is not need additional equations and unknowns from fictitious domains to find approximate solution inside domain Ω.

3 Discretization of the fractional step of convection-diffusion

3.1 Further splitting and discretization of the equation for the first component of velocity

Now we consider the problems (1.9) – (1.11) in turn for $s = 1$ and $s = 2$. First problem has the form:

$$\frac{\partial v_1}{\partial t} - \nu \Delta v_1 + \frac{1}{2}(\mathbf{u}_k^\tau \cdot \nabla) v_1 + \frac{1}{2} div(v_1 \mathbf{u}_k^\tau) = \frac{1}{2} f_1$$

$$\text{in} \quad \Omega \times (t_k, t_{k+1}), \tag{3.1}$$

$$v_1 = g_1 \quad \text{on} \quad \Gamma \times [t_k, t_{k+1}], \tag{3.2}$$

$$v_1(x, y, t_k) = u_{1,k}^\tau (x, y) \quad \text{in} \quad \Omega. \tag{3.3}$$

Once more realize the splitting of this step in the y and x directions. At first we use the initial condition (3.3) in the following form

$$w(x, y, t_k) = a(x, y) \quad \text{on} \quad \bar{\Omega}. \tag{3.4}$$

To simplify the notations, in this section we put

$$a(x, y) = u_{1,k}^\tau (x, y) \quad \text{and} \quad b(x, y) = u_{2,k}^\tau (x, y). \tag{3.5}$$

Then two problems are solved on the segment (t_k, t_{k+1}). The first problem contains the space derivatives only with respect to y:

$$\frac{\partial w}{\partial t} - \nu \frac{\partial^2 w}{\partial y^2} + \frac{1}{2} b \frac{\partial w}{\partial y} + \frac{1}{2} \frac{\partial (bw)}{\partial y} = \frac{1}{4} f_1 \quad \text{in} \quad \Omega \times (t_k, t_{k+1}), \tag{3.6}$$

$$w = g_1 \quad \text{on} \quad \Gamma_y \times [t_k, t_{k+1}], \tag{3.7}$$

where

$$\Gamma_x = \{(x, y) \in \Gamma : (x = 0) \vee (x = 1), \quad y \in [0, 1)\},$$
$$\Gamma_y = \{(x, y) \in \Gamma : x \in [0, 1), \quad (y = 0) \vee (y = 1)\}.$$

Remark 5. It is necessary to pay attention to the modified boundary condition (3.7) in comparison with (3.2). The condition (3.7) follows from (3.2), but it is less restrictive. On the assumption that

$$w = g_1 \quad \text{on} \quad \Gamma \times [t_k, t_{k+1}]$$

instead of (3.7) we should obtain an overdetermined problem. □

The second problem contains the space derivatives only with respect to x:

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} a \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial (au)}{\partial x} = \frac{1}{4} f_1 \quad \text{in} \quad \Omega \times (t_k, t_{k+1}), \tag{3.8}$$

$$u = g_1 \quad \text{on} \quad \Gamma_x \times [t_k, t_{k+1}], \tag{3.9}$$

$$u(x, y, t_k) = w(x, y, t_{k+1}) \text{ on } \bar{\Omega}. \tag{3.10}$$

The solution of this problem is the result of a loop of two fractional steps on the strip $[t_k, t_{k+1}]$:

$$v_1(x, y, t_{k+1}) = u(x, y, t_{k+1}) \quad \text{on} \quad \bar{\Omega}.$$

Now consider the discretization of the problem (3.4), (3.6) – (3.7). The time discretization is achieved by the difference method by means of substitution (2.1). After rearranging the known terms to the right-hand side we obtain the parametric family (with a parameter x) of stationary ordinary differential equations at time level t_{k+1}

$$\frac{1}{\tau} w^{k+1} - \nu \frac{\partial^2 w^{k+1}}{\partial y} + \frac{1}{2} b \frac{\partial w^{k+1}}{\partial y} + \frac{1}{2} \frac{\partial (b w^{k+1})}{\partial y} = \frac{1}{\tau} a + \frac{1}{4} f_1^{k+1} \tag{3.11}$$

in Ω with the boundary condition

$$w^{k+1} = g_1^{k+1} \quad \text{on} \quad \Gamma_y. \tag{3.12}$$

For the space discretization we apply the finite elements method. Therefore we turn to the generalized formulation. Take an arbitrary function $v(x, y)$ which satisfies the condition

$$v = 0 \quad \text{on} \quad \Gamma_y. \tag{3.13}$$

Multiply the equation (3.11) by v and integrate by parts over Ω with application of (3.13). As a result, we obtain the equality

$$\begin{aligned} &\frac{1}{\tau}(w^{k+1}, v)_\Omega + \nu \left(\frac{\partial w^{k+1}}{\partial y}, \frac{\partial v}{\partial y} \right)_\Omega + \frac{1}{2} \left(b \frac{\partial w^{k+1}}{\partial y}, v \right)_\Omega \\ &-\frac{1}{2} \left(b w^{k+1}, \frac{\partial v}{\partial y} \right)_\Omega = \frac{1}{\tau}(a, v)_\Omega + \frac{1}{4}(f_1^{k+1}, v)_\Omega. \end{aligned} \tag{3.14}$$

To approximate this problem, we employ the space H_x introduced in section 2. Besides, denote the following set of boundary nodes by Γ_2^h:

$$\Gamma_2^h = \{ z_{i,0} = (x_i, 0), \; z_{i,n} = (x_i, 1) : i = 0, 1, \ldots, n \}. \tag{3.15}$$

Again theoretically we realize two possibilities: the integration over Ω and the integration over domains with small fictitious additional subdomains to simplify discrete equations. The realization of first approach gives the following Galerkin scheme for the problem (3.12)–(3.14): *find a function* $w^h(x,y) \in H_x$ *which satisfies the boundary condition*

$$w^h = g_1^{k+1} \quad \text{on} \quad \Gamma_2^h \tag{3.16}$$

and the integral relation

$$\begin{aligned} &\frac{1}{\tau}(w^h, v)_\Omega + \nu\left(\frac{\partial w^h}{\partial y}, \frac{\partial v}{\partial y}\right)_\Omega + \frac{1}{2}\left(b\frac{\partial w^h}{\partial y}, v\right)_\Omega \\ &-\frac{1}{2}\left(bw^h, \frac{\partial v}{\partial y}\right)_\Omega = \frac{1}{\tau}(a, v)_\Omega + \frac{1}{4}(f_1^{k+1}, v)_\Omega \end{aligned} \tag{3.17}$$

for an arbitrary function $v \in H_x$ *which satisfies the boundary condition*

$$v = 0 \quad \text{on} \quad \Gamma_2^h. \tag{3.18}$$

We will seek the unknown function w^h in the form

$$w^h(x,y) = \sum_{i=0}^{n}\sum_{j=-1}^{n} w^h_{i,j+1/2}\varphi_{x,i,j+1/2}(x,y). \tag{3.19}$$

The problem (3.16)–(3.18) is equivalent to the system of linear algebraic equations with respect to unknowns $w^h_{i,j+1/2}$. To form the coefficients of this system, we suppose that

$$a \in H_x, \quad b \in H_y. \tag{3.20}$$

This will be ascertained during the final assembling of the discrete time-dependent problem.

To simplify the mass and stiffness matrices, we again use the trapezium quadrature formula (2.12). Consider the inner element $e_{i+1/2,\,j} = (x_i, x_{i+1}) \times (y_{j-1/2}, y_{j+1/2})$. Take the first term in (3.17). Since w^h and v belong to H_x, likewise (2.13) we get

$$\frac{1}{\tau}\int\limits_{e_{i+1/2,\,j}} w^h v\, d\Omega \approx \frac{h^2}{4\tau}\sum_{\pm,\pm}(w^h v)_{i+1/2\pm1/2,\,j\pm1/2}. \tag{3.21}$$

This gives the following impact to the left-hand side of the algebraic bilinear form:

$$[w^h_{i,j-1/2},\ w^h_{i,j+1/2},\ w^h_{i+1,j-1/2},\ w^h_{i+1,j+1/2}]$$
$$\cdot \begin{bmatrix} h^2/4\tau & 0 & 0 & 0 \\ 0 & h^2/4\tau & 0 & 0 \\ 0 & 0 & h^2/4\tau & 0 \\ 0 & 0 & 0 & h^2/4\tau \end{bmatrix} \begin{bmatrix} v_{i,j-1/2} \\ v_{i,j+1/2} \\ v_{i+1,j-1/2} \\ v_{i+1,j+1/2} \end{bmatrix}. \tag{3.22}$$

From the second term of (3.17) we get

$$\nu \int\limits_{e_{i+1/2,j}} \frac{\partial w^h}{\partial y}\frac{\partial v}{\partial y} d\Omega \approx \frac{\nu h^2}{4} \sum_{\pm,\pm} \left(\frac{\partial w^h}{\partial y}\frac{\partial v}{\partial y} \right)_{i+1/2\pm1/2,j\pm1/2}. \tag{3.23}$$

Since

$$\frac{\partial w^h}{\partial y}(x,y) = \frac{w^h(x,y_{j+1/2}) - w^h(x,y_{j-1/2})}{h}$$

and, analogously, $\partial v/\partial y$ is constant in y on $e_{i+1/2,j}$, (3.23) gives the following inpact into the left-hand side of algebraic bilinear form:

$$[w^h_{i,j-1/2},\ w^h_{i,j+1/2},\ w^h_{i+1,j-1/2},\ w^h_{i+1,j+1/2}]$$
$$\cdot \begin{bmatrix} \nu/2 & -\nu/2 & 0 & 0 \\ -\nu/2 & \nu/2 & 0 & 0 \\ 0 & 0 & \nu/2 & -\nu/2 \\ 0 & 0 & -\nu/2 & \nu/2 \end{bmatrix} \begin{bmatrix} v_{i,j-1/2} \\ v_{i,j+1/2} \\ v_{i+1,j-1/2} \\ v_{i+1,j+1/2} \end{bmatrix}. \tag{3.24}$$

We again apply the quadrature formula like (2.12) for the third term of (3.17) taking into consideration that $b \in H_y$:

$$\begin{aligned} \frac{1}{2} \int\limits_{e_{i+1/2,j}} b \frac{\partial w^h}{\partial y} v d\Omega \approx \frac{h^2}{8} b_{i+1/2,j} \Big((v_{i,j-1/2} + v_{i,j+1/2}) \\ \cdot \left(\frac{\partial w^h}{\partial y} \right)_{ij} + (v_{i+1,j-1/2} + v_{i+1,j+1/2}) \left(\frac{\partial w^h}{\partial y} \right)_{i+1,j} \Big). \end{aligned} \tag{3.25}$$

This gives the following impact to the elementary stiffness matrix:

$$[w^h_{i,j-1/2},\ w^h_{i,j+1/2},\ w^h_{i+1,j-1/2},\ w^h_{i+1,j+1/2}]$$

$$\cdot \begin{bmatrix} -\frac{h}{8}b_{i+1/2,j} & -\frac{h}{8}b_{i+1/2,j} & 0 & 0 \\ \frac{h}{8}b_{i+1/2,j} & \frac{h}{8}b_{i+1/2,j} & 0 & 0 \\ 0 & 0 & -\frac{h}{8}b_{i+1/2,j} & -\frac{h}{8}b_{i+1/2,j} \\ 0 & 0 & \frac{h}{8}b_{i+1/2,j} & \frac{h}{8}b_{i+1/2,j} \end{bmatrix} \begin{bmatrix} v_{i,j-1/2} \\ v_{i,j+1/2} \\ v_{i+1,j-1/2} \\ v_{i+1,j+1/2} \end{bmatrix}. \tag{3.26}$$

The similar formulae are valid for the fourth term (with the change of w^h to v and then the multiplication by -1):

$$-\frac{1}{2}\int\limits_{e_{i+1/2,j}} bw^h\frac{\partial v}{\partial y}d\Omega \approx -\frac{h^2}{8}b_{i+1/2,j}\Big((w^h_{i,j-1/2}+w^h_{i,j+1/2}) \tag{3.27}$$

$$\cdot\left(\frac{\partial v}{\partial y}\right)_{ij} + (w^h_{i+1,j-1/2}+v_{i+1,j+1/2})\left(\frac{\partial v}{\partial y}\right)_{i+1,j}\Big).$$

The corresponding impact to the elementary stiffness matrix is

$$[w^h_{i,j-1/2},\ w^h_{i,j+1/2},\ w^h_{i+1,j-1/2},\ w^h_{i+1,j+1/2}]$$

$$\cdot \begin{bmatrix} \frac{h}{8}b_{i+1/2,j} & -\frac{h}{8}b_{i+1/2,j} & 0 & 0 \\ \frac{h}{8}b_{i+1/2,j} & -\frac{h}{8}b_{i+1/2,j} & 0 & 0 \\ 0 & 0 & \frac{h}{8}b_{i+1/2,j} & -\frac{h}{8}b_{i+1/2,j} \\ 0 & 0 & \frac{h}{8}b_{i+1/2,j} & -\frac{h}{8}b_{i+1/2,j} \end{bmatrix} \begin{bmatrix} v_{i,j-1/2} \\ {}_{i,j+1/2} \\ v_{i+1,j-1/2} \\ v_{i+1,j+1/2} \end{bmatrix}. \tag{3.28}$$

At last we consider the right-hand side of (3.17). The quadrature formula (2.12) gives

$$\int\limits_{e_{i+1/2,j}} \left(\frac{1}{\tau}av+\frac{1}{4}f_1^{k+1}v\right)d\Omega \approx \sum_{\pm,\pm}\left(\frac{h^2}{4\tau}av+\frac{h^2}{16}f_1^{k+1}v\right)_{i+1/2\pm1/2,\,j\pm1/2}.$$

This implies the following impact to the right-hand side of the assembled algebraic system:

$$\left[\frac{h^2}{16}\left(f_1^{k+1}+\frac{4}{\tau}a\right)_{i,j-1/2},\quad \frac{h^2}{16}\left(f_1^{k+1}+\frac{4}{\tau}a\right)_{i,j+1/2},\right.$$
$$\left.\frac{h^2}{16}\left(f_1^{k+1}+\frac{4}{\tau}a\right)_{i+1,j-1/2},\quad \frac{h^2}{16}\left(f_1^{k+1}+\frac{4}{\tau}a\right)_{i+1,j+1/2}\right] \tag{3.29}$$
$$\cdot[v_{i,j-1/2},\ v_{i,j+1/2},\ v_{i+1,j-1/2},\ v_{i+1,j+1/2}]^T.$$

Combining (3.22), (3.24), (3.26), and (3.28), we obtain the stiffness matrix of the element $e_{i+1/2,j}$:

$$\begin{bmatrix} \frac{h^2}{4\tau}+\frac{\nu}{2} & -\frac{\nu}{2}-\frac{h}{4}b_{i+1/2,j} & 0 & 0 \\ -\frac{\nu}{2}+\frac{h}{4}b_{i+1/2,j} & \frac{h^2}{4\tau}+\frac{\nu}{2} & 0 & 0 \\ 0 & 0 & \frac{h^2}{4\tau}+\frac{\nu}{2} & -\frac{\nu}{2}-\frac{h}{4}b_{i+1/2,j} \\ 0 & 0 & -\frac{\nu}{2}+\frac{h}{4}b_{i+1/2,j} & \frac{h^2}{4\tau}+\frac{\nu}{2} \end{bmatrix}. \tag{3.30}$$

To study the grid equations further we make its nodal assembly. In order for an arbitrary value of $v_{i,j+1/2}$ to satisfy the equality (3.17), we must equate its coefficients in the left-hand and right-hand sides. At inner nodes four elements $e_{i\pm1/2,j}$ $e_{i\pm1/2,j+1}$ have nonzero coefficients. Summing these coefficients over four elements, equate them in the left-hand and right-hand sides:

$$\left(-\nu-\frac{h}{4}(b_{i-1/2,j}+b_{i+1/2,j})\right)w^h_{i,j-1/2}+\left(\frac{h^2}{\tau}+2\nu\right)w^h_{i,j+1/2}$$
$$+\left(-\nu+\frac{h}{4}(b_{i-1/2,j+1}+b_{i+1/2,j+1})\right)w^h_{i,j+3/2} \tag{3.31}$$
$$=\frac{h^2}{\tau}a_{i,j+1/2}+\frac{h^2}{4}f^{k+1}_{1,i,j+1/2},$$
$$i=1,2,\ldots,n-1;\quad j=1,2,\ldots,n-2.$$

Taking into consideration that $b \in H_y$ we get the shorter form

$$\left(-\nu - \frac{h}{2}b^*_{ij}\right) w^h_{i,j-1/2} + \left(\frac{h^2}{\tau} + 2\nu\right) w^h_{i,j+1/2}$$
$$+ \left(-\nu + \frac{h}{2}b^*_{ij+1}\right) w^h_{i,j+3/2} = \frac{h^2}{\tau} a_{i,j+1/2} + \frac{h^2}{4} f^{k+1}_{1,i,j+1/2}, \qquad (3.32)$$
$$i = 1, \dots, n-1; \quad j = 1, \dots, n-2.$$

From here on the asterisk * warns that this value of function is a linear combination of nodal values, for example,

$$b^*_{ij} = (b_{i-1/2,j} + b_{i+1/2,j})/2.$$

At the boundary nodes of Γ^h_x the assembly is fulfilled over two elements only:

$$\begin{aligned} &\left(-\frac{\nu}{2} - \frac{h}{4}b_{1/2,j}\right) w^h_{0,j-1/2} + \left(\frac{h^2}{2\tau} + \nu\right) w^h_{0,j+1/2} \\ &+ \left(-\frac{\nu}{2} + \frac{h}{4}b_{1/2,j+1}\right) w^h_{0,j+3/2} = \frac{h^2}{2\tau} a_{0,j+1/2} + \frac{h^2}{8} f^{k+1}_{1,0,j+1/2}, \end{aligned} \qquad (3.33)$$

$$\begin{aligned} &\left(-\frac{\nu}{2} - \frac{h}{4}b_{n-1/2,j}\right) w^h_{n,j-1/2} + \left(\frac{h^3}{2\tau} + \nu\right) w^h_{n,j+1/2} \\ &+ \left(-\frac{\nu}{2} + \frac{h}{4}b_{n-1/2,j+1}\right) w^h_{n,j+3/2} = \frac{h^2}{2\tau} a_{n,j+1/2} + \frac{h^2}{8} f^{k+1}_{1,n,j+1/2}, \end{aligned} \qquad (3.34)$$
$$j = 1, \dots, n-2.$$

To close the system of linear algebraic equations, first we amplify it by boundary conditions

$$w^h(x_i, y_j) = g^{k+1}_{ij}, \quad i = 0, 1, \dots, n; \; j = 0, n. \qquad (3.35)$$

To do more laconic the form of these equalities we introduce the grid operators of the local averaging in x and in y:

$$\begin{aligned} u_{\hat{x}}(x) &= (u(x - h/2) + u(x + h/2))/2, \\ u_{\hat{y}}(y) &= (u(y - h/2) + u(y + h/2))/2. \end{aligned} \qquad (3.36)$$

With the help of the designation (3.15) we can write

$$w^h = w^h_{\hat{y}} = g^{k+1}_1 \quad \text{on} \quad \Gamma^h_2. \qquad (3.37)$$

Analogously we get

$$v = v_{\hat{y}} = 0 \quad \text{on} \quad \Gamma_2^h. \tag{3.38}$$

Now consider the stiffness matrix of the elements which are cut by the boundary Γ_y. Let us take $e_{i+1/2,0}$. Taking into consideration the boundary condition (3.38) and the twice as small domain $e_{i+1/2,0} \cap \Omega$ we transform (3.21) into the approximate equality

$$\frac{1}{\tau} \int\limits_{e_{i+1/2,\,0} \cap \Omega} w^h v d\Omega \approx \frac{h^2}{8\tau} \sum_{\pm} (w^h v)_{i+1/2 \pm 1/2,\,1/2}. \tag{3.39}$$

This gives the following impact to the left-hand side of the algebraic bilinear form:

$$[w^h_{i,1/2},\ w^h_{i+1,1/2}] \begin{bmatrix} h^2/8\tau & 0 \\ 0 & h^2/8\tau \end{bmatrix} \begin{bmatrix} v_{i,1/2} \\ v_{i+1,1/2} \end{bmatrix}. \tag{3.40}$$

The quadrature formula (3.23) is transformed to the approximate equality

$$\nu \int\limits_{e_{i+1/2,0} \cap \Omega} \frac{\partial w^h}{\partial y} \frac{\partial v}{\partial y} d\Omega \approx \frac{\nu h^2}{8} \sum_{\pm,\pm} \left(\frac{\partial w^h}{\partial y} \frac{\partial v}{\partial y} \right)_{i+1/2 \pm 1/2,\, 1/4 \pm 1/4}. \tag{3.41}$$

This time

$$\frac{\partial w^h}{\partial y}(x,y) = \frac{w^h(x, y_{1/2}) - w^h(x,0)}{h/2} \quad \text{on } e_{i+1/2,0} \cap \Omega$$

and, analogously, $\partial v / \partial y$ are constant in y, then (3.41) gives the following impact into the algebraic bilinear form:

$$[w^h_{i,0},\ w^h_{i,1/2},\ w^h_{i+1,0},\ w^h_{i+1,1/2}] \begin{bmatrix} -\nu & 0 \\ \nu & 0 \\ 0 & -\nu \\ 0 & \nu \end{bmatrix} \begin{bmatrix} v_{i,1/2} \\ v_{i+1,1/2} \end{bmatrix}. \tag{3.42}$$

For the third term the corresponding quadrature formula gives the approximate equality

$$\begin{aligned} \frac{1}{2} \int\limits_{e_{i+1/2,0} \cap \Omega} b \frac{\partial w^h}{\partial y} v d\Omega \approx \frac{h^2}{16} b^*_{i+1/2,1/4} \\ \cdot \left((v_{i,0} + v_{i,1/2}) \left(\frac{\partial w^h}{\partial y} \right)_{i,0} + (v_{i+1,0} + v_{i+1,1/2}) \left(\frac{\partial w^h}{\partial y} \right)_{i+1,0} \right). \end{aligned} \tag{3.43}$$

This involves the following impact to the algebraic bilinear form:

$$\frac{h}{8} b^*_{i+1/2,1/4} \, [w^h_{i,0},\ w^h_{i,1/2},\ w^h_{i+1,0},\ w^h_{i+1,1/2}] \cdot \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{i,1/2} \\ v_{i+1,1/2} \end{bmatrix}. \tag{3.44}$$

The similar impact is valid for the fourth term:

$$\frac{h}{8} b^*_{i+1/2,1/4} \, [w^h_{i,0},\ w^h_{i,1/2},\ w^h_{i+1,0},\ w^h_{i+1,1/2}] \cdot \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_{i,1/2} \\ v_{i+1,1/2} \end{bmatrix}. \tag{3.45}$$

Combining (3.40), (3.42), (3.44), and (3.45) we get

$$[w^h_{i,0},\ w^h_{i,1/2},\ w^h_{i+1,0},\ w^h_{i+1,1/2}] \cdot \begin{bmatrix} -\frac{h}{4} b^*_{i+1/2,1/4} - \nu & 0 \\ \frac{h^2}{8\tau} + \nu & 0 \\ 0 & -\frac{h^2}{4} b^*_{i+1/2,1/4} - \nu \\ 0 & \frac{h}{8\tau} + \nu \end{bmatrix} \begin{bmatrix} v_{i,1/2} \\ v_{i+1,1/2} \end{bmatrix}. \tag{3.46}$$

Now consider the right-hand side of (3.17). First, the quadrature formula is transformed to the following:

$$\int\limits_{e_{i+1/2,0} \cap \Omega} \left(\frac{1}{\tau} a v + \frac{1}{4} f_1^{k+1} v \right) d\Omega \approx \sum_{\pm} \left(\frac{h^2}{8\tau} a v + \frac{h^2}{32} f_1^{k+1} v \right)_{i+1/2 \pm 1/2,\, 1/2}.$$

Second, due to the boundary condition (3.37) two values $w^h_{i,o}$ and $w^h_{i+1,0}$ are known in (3.46) and their impact has to be rearranged to the right-hand side. This implies that

$$\begin{aligned}&\left[\frac{h^2}{8\tau}a_{i,1/2}+\frac{h^2}{32}f^{k+1}_{1,i,1/2}+\left(\frac{h}{4}b^*_{i+1/2,1/4}+\nu\right)g^{k+1}_{1,i+1/2,0},\right.\\&\left.\frac{h^2}{8\tau}a_{i+1,1/2}+\frac{h^2}{32}f^{k+1}_{1,i+1,1/2}+\left(\frac{h}{4}b^*_{i+1/2,1/4}+\nu\right)g^{k+1}_{1,i+1/2,0}\right]\\&\cdot[v_{i,1/2},\ v_{i+1,1/2}]^T\end{aligned}\tag{3.47}$$

appears in the right-hand side and the rest of (3.46) stays in the left-hand side:

$$[w^h_{i,1/2},\ w^h_{i+1,1/2}]\begin{bmatrix}\dfrac{h^2}{8\tau}+\nu & 0\\ 0 & \dfrac{h^2}{8\tau}+\nu\end{bmatrix}\begin{bmatrix}v_{i,1/2}\\ v_{i+1,1/2}\end{bmatrix}.\tag{3.48}$$

Now assemble the algebraic equations corresponding to $v_{i,1/2}$, $i=1,\dots,n-1$, over 4 elements $e_{i\pm1/2,0}\cap\Omega$ and $e_{i\pm1/2,1}$:

$$\begin{aligned}&\left(\frac{3h^2}{4\tau}+3\nu\right)w^h_{i,1/2}+\left(-\nu+\frac{h}{2}b^*_{i,1}\right)w^h_{i,3/2}\\&=\frac{3h^2}{4\tau}a_{i,1/2}+\frac{3h^2}{16}f^{k+1}_{1,i,1/2}+\left(\frac{h}{2}b^*_{i,1/4}+2\nu\right)g^{k+1}_{1,i,0},\\&\qquad i=1,\dots,n-1.\end{aligned}\tag{3.49}$$

At the boundary node $z_{0,1/2}$ the algebraic equation is assembled over 2 elements $e_{1/2,0}\cap\Omega$ and $e_{1/2,1}$:

$$\begin{aligned}&\left(\frac{3h^2}{8\tau}+\frac{3}{2}\nu\right)w^h_{0,1/2}+\left(-\frac{\nu}{2}+\frac{h}{4}b_{1/2,1}\right)w^h_{0,3/2}\\&=\frac{3h^2}{8\tau}a_{0,1/2}+\frac{3h^2}{32}f^{k+1}_{1,0,1/2}+\left(\frac{h}{4}b^*_{1/2,1/4}+\nu\right)g^{k+1}_{1,0,0}.\end{aligned}\tag{3.50}$$

The similar equation is valid at the node $z_{n,1/2}$:

$$\begin{aligned}&\left(\frac{3h^2}{8\tau}+\frac{3}{2}\nu\right)w^h_{n,1/2}+\left(-\frac{\nu}{2}+\frac{h}{4}b_{n-1/2,1}\right)w^h_{n,3/2}\\&=\frac{3h^2}{8\tau}a_{n,1/2}+\frac{3h^2}{32}f^{k+1}_{1,n,1/2}+\left(\frac{h}{4}b^*_{n-1/2,1/4}+\nu\right)g^{k+1}_{1,n,0}.\end{aligned}\tag{3.51}$$

Without repetition of assembly we write the algebraic equations on the upper part of the boundary Γ_2^h:

$$\begin{gathered}\left(\frac{3h^2}{8\tau}+\frac{3}{2}\nu\right)w^h_{0,n-1/2}+\left(-\frac{\nu}{2}-\frac{h}{4}b_{1/2,n-1}\right)w^h_{0,n-3/2}\\ =\frac{3h^2}{8\tau}a_{0,n-1/2}+\frac{3h^2}{32}f^{k+1}_{1,0,n-1/2}+\left(-\frac{h}{4}b^*_{1/2,n-1/4}+\nu\right)g^{k+1}_{1,0,n};\end{gathered} \tag{3.52}$$

$$\begin{gathered}\left(\frac{3h^2}{4\tau}+3\nu\right)w^h_{i,n-1/2}+\left(-\nu-\frac{h}{2}b^*_{i,n-1}\right)w^h_{i,n-3/2}\\ =\frac{3h^2}{4\tau}a_{i,n-1/2}+\frac{3h^2}{16}f^{k+1}_{1,i,n-1/2}+\left(-\frac{h}{2}b^*_{i,n-1/4}+2\nu\right)g^{k+1}_{1,i,n},\end{gathered} \tag{3.53}$$

$$i=1,\dots,n-1;$$

$$\begin{gathered}\left(\frac{3h^2}{8\tau}+\frac{3}{2}\nu\right)w^h_{n,n-1/2}+\left(-\frac{\nu}{2}-\frac{h}{4}b_{n-1/2,n-1}\right)w^h_{n,n-3/2}\\ =\frac{3h^2}{8\tau}a_{n,n-1/2}+\frac{3h^2}{32}f^{k+1}_{1,n,n-1/2}+\left(-\frac{h}{4}b^*_{n-1/2,n-1/4}+\nu\right)g^{k+1}_{1,n,n}.\end{gathered} \tag{3.54}$$

Let us prove that the obtained system is stable with respect to initial data and a right-hand side. Since the mass matrix is not constant over nodes of Ω_1^h, introduce the weight coefficients

$$\sigma_i=\begin{cases}1 & \text{if } i=1,\dots,n-1,\\ 1/2 & \text{if } i=0,n,\end{cases} \tag{3.55}$$

and

$$\rho_j=\begin{cases}1/4 & \text{if } j=0,n,\\ 3/4 & \text{if } j=1/2,n-1/2,\\ 1 & \text{if } j=3/2,\dots,n-3/2.\end{cases} \tag{3.56}$$

With these weights introduce the norm

$$\|w^h\|^2_{1,\sigma}=h^2\sum_{i=0}^{n}\sigma_i\sum_{j=0}^{n}\rho_{j+1/2}(w^h_{i,j+1/2})^2. \tag{3.57}$$

Theorem 3. *Let the condition*

$$g_1^{k+1}=0 \qquad on \quad \Gamma_y^h \tag{3.58}$$

be valid. Then for the system (3.31)–(3.34), (3.49)–(3.54) *the a priory estimate*

$$\|w^h\|_{1,\sigma}\le\|a\|_{1,\sigma}+\frac{\tau}{4}\|f_1^{k+1}\|_{1,\sigma} \tag{3.59}$$

holds for any grid function $b\in H_y$.

Proof. Multiply each equation of the system mentioned by $w^h_{i,j+1/2}$ with corresponding i, j and sum up over $i = 0, \dots, n,\ j = 0, \dots, n-1$. It is easy to examine that the terms which contain the function b are reciprocally cancelled. For the terms with the multiplier ν we apply the difference analogue of the formula of integration by parts:

$$\begin{aligned}
&\frac{1}{h^2}(w^h_{i,1/2} - w^h_{i,3/2})w^h_{i,1/2} + \frac{1}{h^2}(w^h_{i,n-1/2} - w^h_{i,n-3/2})w^h_{i,n-1/2} \\
&+\frac{1}{h^2}\sum_{j=1}^{n-2}(-w^h_{i,j-1/2} + 2w^h_{i,j+1/2} - w^h_{i,j+3/2})w^h_{i,j+1/2} \\
&= \frac{1}{h^2}\sum_{j=1}^{n-1}(w^h_{i,j+1/2} - w^h_{i,j-1/2})^2.
\end{aligned} \tag{3.60}$$

As a result of this computations we get the inequality

$$\begin{aligned}
&\sum_{i=0}^{n}\sigma_i\left(\frac{h^2}{\tau}\sum_{j=0}^{n-1}\rho_{j+1/2}(w^h_{i,j+1/2})^2 + \nu\sum_{j=1}^{n-1}(w^h_{i,j+1/2} - w^h_{i,j-1/2})^2\right) \\
&\leq \frac{h^2}{\tau}\sum_{i=0}^{n-1}\sigma_i\sum_{j=0}^{n-1}\rho_{j+1/2}a_{i,j+1/2}w^h_{i,j+1/2} \\
&+\frac{h^2}{4}\sum_{i=0}^{n-1}\sigma_i\sum_{j=0}^{n-1}\rho_{j+1/2}f^{k+1}_{1,i,j+1/2}w^h_{i,j+1/2}.
\end{aligned} \tag{3.61}$$

In the left-hand side we drop the second positive sum while in the right-hand side we apply the Cauchy-Bunyakovski inequality:

$$\frac{1}{\tau}\|w^h\|^2_{1,\sigma} \leq \frac{1}{\tau}\|a\|_{1,\sigma}\|w^h\|_{1,\sigma} + \frac{1}{4}\|f^{k+1}\|_{1,\sigma}\|w^h\|_{1,\sigma}.$$

If $\|w^h\|_{1,\sigma} = 0$ then the inequality (3.59) is evident. In the other case we divide both sides by the positive expression $\|w^h\|_{1,\sigma}/\tau$ and get (3.59). □

Now consider the discretization of the problem (3.8) – (3.10). The time discretization is achieved by means of the substitution (2.1). After rearranging the term known due to (3.10) to the right-hand side we obtain the parametric family (with the parameter y) of stationary ordinary differential equations at time level t_{k+1}:

$$\frac{1}{\tau}u^{k+1} - \nu\frac{\partial^2 u^{k+1}}{\partial x^2} + \frac{1}{2}a\frac{\partial u^{k+1}}{\partial x} + \frac{1}{2}\frac{\partial(au^{k+1})}{\partial x} = \frac{1}{\tau}w^{k+1} + \frac{1}{4}f_1^{k+1} \text{ in } \Omega \tag{3.62}$$

with the boundary condition

$$u^{k+1} = g_1^{k+1} \quad \text{on} \quad \Gamma_x. \tag{3.63}$$

For the space discretization we turn to the generalized formulation. To do it we take an arbitrary function $v(x,y)$ which satisfies the condition

$$v = 0 \quad \text{on} \quad \Gamma_x. \tag{3.64}$$

Multiply the equation (3.62) by v and integrate by parts over Ω with the help of (3.64). As a result, we obtain

$$\begin{aligned}&\frac{1}{\tau}(u^{k+1}, v)_\Omega + \nu \left(\frac{\partial u^{k+1}}{\partial x}, \frac{\partial v}{\partial x}\right)_\Omega + \frac{1}{2}\left(a\frac{\partial u^{k+1}}{\partial x}, v\right)_\Omega \\ &-\frac{1}{2}\left(au^{k+1}, \frac{\partial v}{\partial x}\right)_\Omega = \frac{1}{\tau}(w^{k+1}, v)_\Omega + \frac{1}{4}(f_1^{k+1}, v)_\Omega.\end{aligned} \tag{3.65}$$

To approximate this problem, we again employ the space H_x introduced in section 2. As a result, we obtain the following Galerkin problem: *find a function $u^h(x,y) \in H_x$ which satisfies the boundary condition*

$$u^h = g_1^{k+1} \quad on \quad \Gamma_x^h \tag{3.66}$$

and the integral relation

$$\begin{aligned}&\frac{1}{\tau}(u^h, v)_\Omega + \nu \left(\frac{\partial u^h}{\partial x}, \frac{\partial v}{\partial x}\right)_\Omega + \frac{1}{2}\left(a\frac{\partial u^h}{\partial x}, v\right)_\Omega \\ &-\frac{1}{2}\left(au^h, \frac{\partial v}{\partial x}\right)_\Omega = \frac{1}{\tau}(w^h, v)_\Omega + \frac{1}{4}(f_1^{k+1}, v)_\Omega\end{aligned} \tag{3.67}$$

for an arbitrary function $v \in H_x$ which satisfies the boundary condition

$$v = 0 \qquad on \quad \Gamma_x^h. \tag{3.68}$$

It should be noted that in the right-hand side we replace the function w^{k+1} by its approximation $w^h \in H_x$ obtained by solving the problem (3.16)–(3.18).

We will seek the unknown function u^h in the form

$$u^h(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-1} u^h_{i,j+1/2}\varphi_{x,i,j+1/2}(x,y). \tag{3.69}$$

Then the problem (3.67) – (3.69) is equivalent to the system of linear algebraic equations with respect to the unknowns $u^h_{i,j+1/2}$. Using the trapezium quadrature formula, by analogy with computations (3.21) – (3.30) we obtain the following stiffness matrix of the element $e_{i+1/2,j}$:

$$\begin{bmatrix} \dfrac{h^2}{4\tau}+\dfrac{\nu}{2} & 0 & -\dfrac{h}{4}a^*_{i+\frac12,j-\frac12}-\dfrac{\nu}{2} & 0 \\ 0 & \dfrac{h^2}{4\tau}+\dfrac{\nu}{2} & 0 & -\dfrac{h}{4}a^*_{i+\frac12,j+\frac12}-\dfrac{\nu}{2} \\ \dfrac{h}{4}a^*_{i+\frac12,j-\frac12}-\dfrac{\nu}{2} & 0 & \dfrac{h^2}{4\tau}+\dfrac{\nu}{2} & 0 \\ 0 & \dfrac{h}{4}a^*_{i+1/2,j+1/2}-\dfrac{\nu}{2} & 0 & \dfrac{h^2}{4\tau}+\dfrac{\nu}{2} \end{bmatrix}, \tag{3.70}$$

$$i=0,\dots,n-1; \quad j=1,\dots,n-1.$$

The impact of $e_{i+1/2,j}$ into the right-hand side of (3.67) is

$$\sum_{\pm,\pm}\left(\left(\frac{h^2}{16}f_1^{k+1}+\frac{h^2}{4\tau}w^h\right)v\right)_{i+1/2\pm1/2,j\pm1/2}. \tag{3.71}$$

With the help of these impacts first we assemble grid equations at inner nodes $z_{i,j+1/2}$. In order for an arbitrary value $v_{i,j+1/2}$ to satisfy the equality (3.67) we must equate its coefficients in the left-hand and right-hand sides. Four elements $e_{i\pm1/2,j}$, $e_{i\pm1/2,j+1}$ have nonzero coefficients. Summing these coefficients over these elements, equate then:

$$\begin{aligned} &\left(-\nu-\frac{h}{2}a^*_{i-1/2,j+1/2}\right)u^h_{i-1,j+1/2}+\left(\frac{h^2}{\tau}+2\nu\right)u^h_{i,j+1/2} \\ &+\left(-\nu+\frac{h}{2}a^*_{i+1/2,j+1/2}\right)u^h_{i+1,j+1/2}=\frac{h^2}{\tau}w^h_{i,j+1/2}+\frac{h^2}{4}f^{k+1}_{1,i,j+1/2}, \\ &\qquad i=1,\dots,n-1;\ j=1,\dots,n-2. \end{aligned} \tag{3.72}$$

To close the system of linear algebraic equations along these lines we amplify it by the boundary conditions (3.66).

Now consider the stiffness matrix of the elements which are cut by the boundary Γ_y, for example, $e_{i+1/2,0}$. We use the following quadrature formulae:

$$\frac{1}{\tau}\int\limits_{e_{i+1/2,0}\cap\Omega} u^h v d\Omega \approx \frac{h^2}{8\tau}\sum_{\pm,\pm}(u^h v)_{i+1/2\pm1/2,1/4\pm1/4}, \tag{3.73}$$

$$\nu\int\limits_{e_{i+1/2,0}\cap\Omega} \frac{\partial u^h}{\partial x}\frac{\partial v}{\partial x} d\Omega \approx \frac{\nu h^2}{4}\sum_{\pm}\left(\frac{\partial u^h}{\partial x}\frac{\partial v}{\partial x}\right)_{i+1/2,1/4\pm1/4}, \tag{3.74}$$

$$\frac{1}{2}\int\limits_{e_{i+1/2,0}\cap\Omega} a\frac{\partial u^h}{\partial x} v d\Omega \tag{3.75}$$

$$\approx \frac{h^2}{16}\sum_{\pm}((av)_{i,1/4\pm1/4} + (av)_{i+1,1/4\pm1/4})\left(\frac{\partial u^h}{\partial x}\right)_{i+1/2,1/4\pm1/4},$$

$$-\frac{1}{2}\int\limits_{e_{i+1/2,0}\cap\Omega} a u^h \frac{\partial v}{\partial x} d\Omega \tag{3.76}$$

$$\approx \frac{h^2}{16}\sum_{\pm}\left((au^h)_{i,1/4\pm1/4} + (au^h)_{i+1,1/4\pm1/4}\right)\left(\frac{\partial v}{\partial x}\right)_{i+1/2,1/4\pm1/4}.$$

Combine these impacts and write the result in the form of the stiffness matrix:

$$[u^h_{i,0},\ u^h_{i,1/2},\ u^h_{i+1,0},\ u^h_{i+1,1/2}] \tag{3.77}$$

$$\cdot\begin{bmatrix} \frac{h^2}{8\tau}+\frac{\nu}{4} & 0 & -\frac{\nu}{4}-\frac{h}{8}a^*_{i+1/2,0} & 0 \\ 0 & \frac{h^2}{8\tau}+\frac{\nu}{4} & 0 & -\frac{\nu}{4}-\frac{h}{8}a^*_{i+1/2,1/2} \\ -\frac{\nu}{4}+\frac{h}{8}a^*_{i+1/2,0} & 0 & \frac{h^2}{8\tau}+\frac{\nu}{4} & 0 \\ 0 & -\frac{\nu}{4}+\frac{h}{8}a^*_{i+1/2,1/2} & 0 & \frac{h^2}{8\tau}+\frac{\nu}{4} \end{bmatrix}$$

$$\cdot[v_{i,0},\ v_{i,1/2},\ v_{i+1,0},\ v_{i+1,1/2}]^T.$$

The impact of $e_{i+1/2,0}\cap\Omega$ into the right-hand side of (3.67) is

$$\sum_{\pm,\pm}\left(\left(\frac{h^2}{32}f_1^{k+1}+\frac{h^2}{8\tau}w^h\right)v\right)_{i+1/2\pm1/2,1/4\pm1/4}. \tag{3.78}$$

Now assemble the algebraic equations corresponding to $v_{i,1/2}$, $i=1,\ldots,n-1$, over 4 elements $e_{i\pm1/2,0}\cap\Omega$ and $e_{i\pm1/2,1}$:

$$\left(-\frac{3\nu}{4}-\frac{3h}{8}a^*_{i-1/2,1/2}\right)u^h_{i-1,1/2}+\left(\frac{3h^2}{4\tau}+\frac{3\nu}{2}\right)u^h_{i,1/2}$$
$$+\left(-\frac{3\nu}{4}+\frac{3h}{8}a^*_{i+1/2,1/2}\right)u^h_{i+1,1/2}=\frac{3h^2}{4\tau}w^h_{i,1/2}+\frac{3h^2}{16}f^{k+1}_{1,i,1/2},\quad (3.79)$$
$$i=1,\dots,n-1.$$

And finally we assemble the algebraic equations corresponding to $v_{i,0}$, $i=1,\dots,n-1$, over 2 elements $e_{i\pm1/2,0}\cap\Omega$:

$$\left(-\frac{\nu}{4}-\frac{h}{8}a^*_{i-1/2,0}\right)u^h_{i-1,0}+\left(\frac{h^2}{4\tau}+\frac{\nu}{2}\right)u^h_{i,0}$$
$$+\left(-\frac{\nu}{4}+\frac{h}{8}a^*_{i+1/2,0}\right)u^h_{i+1,0}=\frac{h^2}{4\tau}w^h_{i,0}+\frac{h^2}{16}f^{k+1}_{1,i,0},\quad (3.80)$$
$$i=1,\dots,n-1.$$

Similar equations arise near the upper part of the boundary Γ_y:

$$\left(-\frac{3\nu}{4}-\frac{3h}{8}a^*_{i-1/2,n-1/2}\right)u^h_{i-1,n-1/2}$$
$$+\left(\frac{3h^2}{4\tau}+\frac{3\nu}{2}\right)u^h_{i,n-1/2}+\left(-\frac{3\nu}{4}+\frac{3h}{8}a^*_{i+1/2,n-1/2}\right)u^h_{i+1,n-1/2}\quad (3.81)$$
$$=\frac{3h^2}{4\tau}w^h_{i,n-1/2}+\frac{3h^2}{16}f^{k+1}_{1,i,n-1/2},$$

and

$$\left(-\frac{\nu}{4}-\frac{h}{8}a^*_{i-1/2,n}\right)u^h_{i-1,n}+\left(\frac{h^2}{4\tau}+\frac{\nu}{2}\right)u^h_{i,n}$$
$$+\left(-\frac{\nu}{4}+\frac{h}{8}a^*_{i+1/2,n}\right)u^h_{i+1,n}=\frac{h^2}{4\tau}w^h_{i,n}+\frac{h^2}{16}f^{k+1}_{1,i,n},\quad (3.82)$$
$$i=1,\dots,n-1.$$

By analogy with the proof of Theorem 3 we obtain the stability of the system (3.66), (3.72), (3.79) – (3.82) with respect to initial data and a right-hand side, which we describe without substantiation. For this purpose we introduce the norm

$$\|u^h\|_{1,\rho}^2 = h^2 \sum_{i=1}^{n-1} \left(\sum_{j=0}^{n-1} \rho_{j+1/2} (u^h_{i,j+1/2})^2 + \rho_0 (u^h_{i,0})^2 + \rho_n (u^h_{i,n})^2 \right). \quad (3.83)$$

Theorem 4. *When the condition*

$$g_1^{k+1} = 0 \quad on \quad \Gamma_x^h \quad (3.84)$$

is valid, for the system (3.66), (3.72), (3.79)–(3.82) *the a priori estimate*

$$\|u^h\|_{1,\rho} \le \|w^h\|_{1,\rho} + \frac{\tau}{4} \|f_1^{k+1}\|_{1,\rho} \quad (3.85)$$

holds for any grid function $a \in H_x$. □

3.2 Splitting and discretization of the equation for the second component of velocity

Now we consider the problem (1.9) – (1.11) for the second component of velocity:

$$\frac{\partial v_2}{\partial t} - \nu \Delta v_2 + \frac{1}{2} (\mathbf{u}_k^\tau \cdot \nabla) v_2 + \frac{1}{2} div(v_2 \mathbf{u}_k^\tau) = \frac{1}{2} f_2$$
$$\text{in} \quad \Omega \times (t_k, t_{k+1}), \quad (3.86)$$
$$v_2 = g_2 \quad \text{on} \quad \Gamma \times (t_k, t_{k+1}), \quad (3.87)$$
$$v_2(x, y, t_k) = u^\tau_{2,k}(x, y) \quad \text{in} \quad \Omega. \quad (3.88)$$

Realize the further splitting of this fractional step in x- and y-directions:

$$\frac{\partial \bar{u}}{\partial t} - \nu \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{1}{2} a \frac{\partial \bar{u}}{\partial x} + \frac{1}{2} \frac{\partial (a \bar{u})}{\partial x} = \frac{1}{4} f_2$$
$$\text{in} \quad \Omega \times (t_k, t_{k+1}), \quad (3.89)$$
$$\bar{u} = g_2 \quad \text{on} \quad \Gamma_x \times (t_k, t_{k+1}), \quad (3.90)$$
$$\bar{u}(x, y, t_k) = b(x, y) \quad \text{on} \quad \bar{\Omega}; \quad (3.91)$$

and

$$\frac{\partial \bar{w}}{\partial t} - \nu \frac{\partial^2 \bar{w}}{\partial y^2} + \frac{1}{2} b \frac{\partial \bar{w}}{\partial y} + \frac{1}{2} \frac{\partial (b \bar{w})}{\partial y} = \frac{1}{4} f_2$$
$$\text{in} \quad \Omega \times (t_k, t_{k+1}], \quad (3.92)$$
$$\bar{w} = g_2 \quad \text{on} \quad \Gamma_y \times (t_k, t_{k+1}), \quad (3.93)$$
$$\bar{w}(x, y, t_k) = \bar{u}(x, y, t_{k+1}) \quad \text{on} \quad \bar{\Omega}. \quad (3.94)$$

After solving both problems we get

$$v_2(x,y,t_{k+1}) = \bar{w}(x,y,t_{k+1}) \quad \text{on} \quad \bar{\Omega}. \tag{3.95}$$

The equations (3.89) and (3.92) are identical to the equations (3.8) and (3.6) respectively, while the problems differ only in right-hand side. Therefore we do not repeat detailed computations for the discretization of two new problems and derive the final grid formulation. The difference in the discretization of new problems consists in the use of different (geometrically shifted) subspaces for u and $\bar{u}, w$ and $\bar{w}$. From the geometric point of view the problems (3.6)–(3.7) and (3.89)–(3.91) are more contiguous within the change of y to x, b to a, which we just used.

First we write the Galerkin problem for the problem (3.89)–(3.91): *find a function $\bar{u}^h(x,y) \in H_y$ which satisfies the boundary condition*

$$\bar{u}^h = g_2^{k+1} \qquad on \qquad \Gamma_1^h \tag{3.96}$$

and the integral relation

$$\begin{aligned} &\frac{1}{\tau}(\bar{u}^h, v)_\Omega + \nu\left(\frac{\partial \bar{u}^h}{\partial x}, \frac{\partial v}{\partial x}\right)_\Omega + \frac{1}{2}\left(a\frac{\partial \bar{u}^h}{\partial x}, v\right)_\Omega \\ &-\frac{1}{2}\left(a\bar{u}^h, \frac{\partial v}{\partial x}\right)_\Omega = \frac{1}{\tau}(b, v)_\Omega + \frac{1}{4}(f_2^{k+1}, v)_\Omega \end{aligned} \tag{3.97}$$

for an arbitrary function $v \in H_y$ which satisfies the boundary condition

$$v = 0 \quad on \quad \Gamma_1^h. \tag{3.98}$$

We will seek the unknown function $\bar{u}^h$ in the form

$$\bar{u}^h(x,y) = \sum_{i=-1}^{n}\sum_{j=0}^{n} \bar{u}^h_{i+1/2,j}\varphi_{y,i+1/2,j}(x,y). \tag{3.99}$$

The problem (3.96)–(3.98) is equivalent to the system of linear algebraic equations with respect to unknowns $\bar{u}^h_{i+1/2,j}$:

$$\begin{aligned} &\left(-\nu - \frac{h}{2}a^*_{i,j}\right)\bar{u}^h_{i-1/2,j} + \left(\frac{h^2}{\tau} + 2\nu\right)\bar{u}^h_{i+1/2,j} \\ &+\left(-\nu + \frac{h}{2}a^*_{i+1,j}\right)\bar{u}^h_{i+3/2,j} = \frac{h^2}{\tau}b_{i+1/2,j} + \frac{h^2}{4}f^{k+1}_{2,i+1/2,j}, \\ &\quad i = 1,\dots,n-2; \quad j = 1,\dots,n-1. \end{aligned} \tag{3.100}$$

At the boundary nodes of Γ_y^h the assembly is fulfilled over 2 elements only:

$$\begin{aligned}&\left(-\frac{\nu}{2}a_{i,1/2}\right)\bar{u}^h_{i-1/2,0}+\left(\frac{h^2}{2\tau}+\nu\right)\bar{u}^h_{i+1/2,0}\\&+\left(-\frac{\nu}{2}+\frac{h}{4}a_{i+1,1/2}\right)\bar{u}^h_{i+3/2,0}=\frac{h^2}{2\tau}b_{i+1/2,0}+\frac{h^2}{8}f^{k+1}_{2,i+1/2,0},\end{aligned} \tag{3.101}$$

$$\begin{aligned}&\left(-\frac{\nu}{2}-\frac{h}{4}a_{i,n-1/2}\right)\bar{u}^h_{i-1/2,n}+\left(\frac{h^2}{2\tau}+\nu\right)\bar{u}^h_{i+1/2,n}\\&+\left(-\frac{\nu}{2}+\frac{h}{4}a_{i+1,n-1/2}\right)\bar{u}^h_{i+3/2,n}=\frac{h^2}{2\tau}a_{i+1/2,n}+\frac{h^2}{8}f^{k+1}_{2,i+1/2,n},\\&\quad i=1,\dots,n-2.\end{aligned} \tag{3.102}$$

Near the boundary Γ_1^h the algebraic equations are somewhat modified due to twice as small elements $e_{0,j+1/2}\cap\Omega$ (instead of $e_{0,j+1/2}$):

$$\begin{aligned}&\left(\frac{3h^2}{4\tau}+3\nu\right)\bar{u}^h_{1/2,j}+\left(-\nu+\frac{h}{2}a^*_{1,j}\right)\bar{u}^h_{3/2,j}\\&=\frac{3h^2}{4\tau}b_{1/2,j}+\frac{3h^2}{16}f^{k+1}_{1,1/2,j}+\left(\frac{h}{2}a^*_{1/4,j}+2\nu\right)g^{k+1}_{2,0,j},\\&\quad j=1,\dots,n-1.\end{aligned} \tag{3.103}$$

Note again that $a\in H_x$ and its values at intermediate points are the corresponding linear combinations of nodal values. For example,

$$\begin{aligned}&a^*_{1,j}=(a_{1,j-1/2}+a_{1,j+1/2})/2,\\&a^*_{1/4,j+1/2}=(3a_{0,j+1/2}+a_{1,j+1/2})/4,\\&a^*_{1/4,j}=(3a_{0,j-1/2}+a_{1,j-1/2}+3a_{0,j+1/2}+a_{1,j+1/2})/8.\end{aligned} \tag{3.104}$$

At the boundary nodes $z_{1/2,0}$ and $z_{1/2,n}$ we get

$$\left(\frac{3h^2}{8\tau}+\frac{3}{2}\nu\right)\bar{u}^h_{1/2,0}+\left(-\frac{\nu}{2}+\frac{h}{4}a_{1,1/2}\right)\bar{u}^h_{3/2,0} \tag{3.105}$$

$$=\frac{3h^2}{8\tau}b_{1/2,0}+\frac{3h^2}{32}f^{k+1}_{2,1/2,0}+\left(\frac{h}{4}a^*_{1/4,1/2}+\nu\right)g^{k+1}_{2,0,0},$$

$$\left(\frac{3h^2}{8\tau}+\frac{3}{2}\nu\right)\bar{u}^h_{1/2,n}+\left(-\frac{\nu}{2}+\frac{h}{4}a_{1,n-1/2}\right)\bar{u}^h_{3/2,n} \tag{3.106}$$

$$=\frac{3h^2}{8\tau}b_{1/2,n}+\frac{3h^2}{32}f^{k+1}_{2,1/2,n}+\left(\frac{h}{4}a^*_{1/4,n-1/2}+\nu\right)g^{k+1}_{2,0,n}.$$

Similar equations are valid at the nodes $z_{n-1/2,0}$, $z_{n-1,j}$ for $j=1,\dots,n-1$, and $z_{n-1/2,n}$:

$$\left(\frac{3h^2}{8\tau}+\frac{3}{2}\nu\right)\bar{u}^h_{n-1/2,0}+\left(-\frac{\nu}{2}-\frac{h}{4}a_{n-1,1/2}\right)\bar{u}^h_{n-3/2,0} \tag{3.107}$$

$$=\frac{3h^2}{8\tau}b_{n-1/2,0}+\frac{3h^2}{32}f^{k+1}_{2,n-1/2,0}+\left(-\frac{h}{4}a^*_{n-1/4,1/2}+\nu\right)g^{k+1}_{2,n,0};$$

$$\left(\frac{3h^2}{4\tau}+3\nu\right)\bar{u}^h_{n-1/2,j}+\left(-\nu-\frac{h}{2}a^*_{n-1,j}\right)\bar{u}^h_{n-3/2,j} \tag{3.108}$$

$$=\frac{3h^2}{4\tau}b_{n-1/2,j}+\frac{3h^2}{16}f^{k+1}_{2,n-1/2,j}+\left(-\frac{h}{2}a^*_{n-1/4,j}+2\nu\right)g^{k+1}_{2,n,j},$$

$$j=1,\dots,n-1;$$

$$\left(\frac{3h^2}{8\tau}+\frac{3}{2}\nu\right)\bar{u}^h_{n-1/2,n}+\left(-\frac{\nu}{2}-\frac{h}{4}a_{n-1,n-1/2}\right)\bar{u}^h_{n-3/2,n} \tag{3.109}$$

$$=\frac{3h^2}{8\tau}b_{n-1/2,n}+\frac{3h^2}{32}f^{k+1}_{2,n-1/2,n}+\left(-\frac{h}{4}a^*_{n-1/4,n-1/2}+\nu\right)g^{k+1}_{2,n,n}.$$

With the help of the notations (3.55), (3.56) introduce the norm

$$\|\bar{u}^h\|^2_{2,\sigma}=h^2\sum_{i=0}^{n-1}\rho_{i+1/2}\sum_{j=0}^{n}\sigma_j(\bar{u}^h_{i+1/2,j})^2. \tag{3.110}$$

Theorem 5. *Let the condition*

$$g^{k+1}_2=0 \quad on \quad \Gamma^h_x \tag{3.111}$$

be valid. Then for the system (3.100)–(3.103), (3.105)–(3.109) *the a priory estimate*

$$\|\bar{u}^h\|_{2,\sigma} \le \|b\|_{2,\sigma} + \frac{\tau}{4}\|f_2^{k+1}\|_{2,\sigma} \tag{3.112}$$

holds for any grid function $a \in H_x$. □

Finally, consider the last problem (3.92)–(3.94). It is contiguous to the problem (3.8)–(3.10) within the change of x to y, a to b. The Galerkin problem for implicit discretization in time looks as follows: *find a function* $\bar{w}^h(x,y) \in H_y$ *which satisfies the boundary condition*

$$\bar{w}^h = g_2^{k+1} \quad on \quad \Gamma_y^h \tag{3.113}$$

and the integral relation

$$\begin{aligned} &\frac{1}{\tau}(\bar{w}^h, v)_\Omega + \nu\left(\frac{\partial \bar{w}^h}{\partial y}, \frac{\partial v}{\partial y}\right)_\Omega + \frac{1}{2}\left(\frac{\partial \bar{w}^h}{\partial y}, v\right)_\Omega \\ &-\frac{1}{2}\left(b\bar{w}^h, \frac{\partial v}{\partial y}\right)_\Omega = \frac{1}{\tau}(\bar{u}^h, v)_\Omega + \frac{1}{4}\left(f_2^{k+1}, v\right)_\Omega \end{aligned} \tag{3.114}$$

for an arbitrary function $v \in H_y$ *which satisfies the boundary condition*

$$v = 0 \quad on \quad \Gamma_x^h. \tag{3.115}$$

Note, that in the right-hand side we replace the function $\bar{u}$ by its approximation $\bar{u}^h \in H_y$ obtained by solving the problem (3.99)–(3.103), (3.105)–(3.109). We will seek the unknown function $\bar{w}^h$ in the form

$$\bar{w}^h(x,y) = \sum_{i=0}^{n-1}\sum_{j=0}^{n} \bar{w}^h_{i+1/2,j}\varphi_{y,i+1/2,j}(x,y). \tag{3.116}$$

The problem (3.113)–(3.115) is equivalent to the system of linear algebraic equations with respect to unknowns $\bar{w}^h_{i+1/2,j}$:

$$\begin{aligned} &\left(-\nu - \frac{h}{2}b^*_{i+1/2,j-1/2}\right)\bar{w}^h_{i+1/2,j-1} + \left(\frac{h^2}{\tau} + 2\nu\right)\bar{w}^h_{i+1/2,j} \\ &+\left(-\nu + \frac{h}{2}b^*_{i+1/2,j+1/2}\right)\bar{w}^h_{i+1/2,j+1} = \frac{h^2}{\tau}\bar{u}^h_{i+1/2,j} + \frac{h^2}{4}f^{k+1}_{2,i+1/2,j}, \\ &\quad i = 1, \dots, n-2; \end{aligned} \tag{3.117}$$

$$\left(-\frac{3\nu}{4}-\frac{3h}{8}b^*_{1/2,j-1/2}\right)\bar{w}^h_{1/2,j-1}+\left(\frac{3h^2}{4\tau}+\frac{3\nu}{2}\right)\bar{w}^h_{1/2,j}$$
$$+\left(-\frac{3\nu}{4}+\frac{3h}{8}b^*_{1/2,j+1/2}\right)\bar{w}^h_{1/2,j+1}=\frac{3h^2}{4\tau}\bar{u}^h_{1/2,j}+\frac{3h^2}{16}f^{k+1}_{2,1/2,j}; \tag{3.118}$$

$$\left(-\frac{\nu}{4}-\frac{h}{8}b_{0,j-1/2}\right)\bar{w}^h_{0,j-1}+\left(\frac{h^2}{4\tau}+\frac{\nu}{2}\right)\bar{w}^h_{0,j}$$
$$+\left(-\frac{\nu}{4}+\frac{h}{8}b_{0,j+1/2}\right)\bar{w}^h_{0,j+1}=\frac{h^2}{4\tau}\bar{u}^h_{0,j}+\frac{h^2}{16}f^{k+1}_{2,0,j}; \tag{3.119}$$

$$\left(-\frac{3\nu}{4}-\frac{3h}{8}b^*_{n-1/2,j-1/2}\right)\bar{w}^h_{n-1/2,j-1}+\left(\frac{3h^2}{4\tau}+\frac{3\nu}{2}\right)\bar{w}^h_{n-1/2,j}$$
$$+\left(-\frac{3\nu}{4}+\frac{3h}{8}b^*_{n-1/2,j+1/2}\right)\bar{w}^h_{n-1/2,j+1}=\frac{3h^2}{4\tau}\bar{u}^h_{n-1/2,j}+\frac{3h^2}{16}f^{k+1}_{2,n-1/2,j}; \tag{3.120}$$

$$\left(-\frac{\nu}{4}-\frac{h}{8}b_{n,j-1/2}\right)\bar{w}^h_{n,j-1}+\left(\frac{h^2}{4\tau}+\frac{\nu}{2}\right)\bar{w}^h_{n,j}$$
$$+\left(-\frac{\nu}{4}+\frac{h}{8}b_{n,j+1/2}\right)\bar{w}^h_{n,j+1}=\frac{h^2}{4\tau}\bar{u}^h_{n,j}+\frac{h^2}{16}f^{k+1}_{2,n,j}; \tag{3.121}$$

$$j=1,\dots,n-1.$$

The stability of this system is substantiated in the norm

$$\|\bar{w}\|^2_{2,\rho}=h^2\sum_{j=1}^{n-1}\left(\sum_{i=0}^{n-1}\rho_{i+1/2}(\bar{w}^h_{i+1/2,j})^2+\rho_0(\bar{w}^h_{0,j})^2+\rho_n(\bar{w}_{n,j})^2\right). \tag{3.122}$$

Theorem 6. *When the condition*

$$g_2^{k+1}=0 \quad on \quad \Gamma^h_y \tag{3.123}$$

is valid, for the system (3.113), (3.117)–(3.121) *the a priori estimate*

$$\|\bar{w}^h\|_{2,\rho}\le\|\bar{u}^h\|_{2,\rho}+\frac{\tau}{4}\|f_2^{k+1}\|_{2,\rho} \tag{3.124}$$

holds for any $b\in H_y$. □

3.3 Integration with the help of small fictitious domains for uniformity of equations

To realize the approach with small fictitious domains we first consider extended domain $\Omega_1=(0,\ 1)\times(-h/2,\ 1+h/2)$ and prolong the equation

(3.6) by smooth way into additional strips $\Omega_1 \setminus \Omega$ through boundary Γ_y using Taylor expansions of functions in left-hand side of (3.6). Recompute right-hand side of (3.6) with the help of extended functions in fictitious domains, we get equation (3.11) to be valid in extended domain Ω_1.

With these extensions we obtain the following Galerkin formulation instead of (3.17): *find $w^h \in H_x$ which satisfies the boundary condition*

$$w^h_{i,j} = g^{k+1}_{1,i,j}, \; i = 0, \dots, n, \; j = 0, n, \tag{3.125}$$

and the integral relation

$$\begin{aligned} &\frac{1}{\tau}(w^h, v)_{\Omega_1} + \nu \left(\frac{\partial w^h}{\partial y}, \frac{\partial v}{\partial y} \right)_{\Omega_1} + \frac{1}{2}\left(b\frac{\partial \bar{w}^h}{\partial y}, v \right)_{\Omega_1} \\ &-\frac{1}{2}\left(bw^h, \frac{\partial v}{\partial y} \right)_{\Omega_1} = \frac{1}{\tau}(a, v)_{\Omega_1} + \frac{1}{4}\left(f^{k+1}_1, v \right)_{\Omega_1} \end{aligned} \tag{3.126}$$

for an arbitrary function $v \in H_x$ which satisfies the boundary condition

$$v_{i,j} = 0, \quad i = 0, \dots, n, \quad j = 0, n. \tag{3.127}$$

Repeating considerations (3.21)–(3.30) on the extended domain Ω_1 we get the same elemental stiffness matrix even for elements which are cut by boundary Γ_y. Take for example element $e_{i+1/2,0}$ and its stiffness matrix

$$[w^h_{i,-1/2}, \; w^h_{i,1/2}, \; w^h_{i+1,-1/2}, \; w^h_{i+1,1/2}]$$

$$\cdot \begin{bmatrix} \frac{h^2}{4\tau} + \frac{\nu}{2} & -\frac{\nu}{2} - \frac{h}{4}b_{i+1/2,0} & 0 & 0 \\ -\frac{\nu}{2} - \frac{h}{4}b_{i+1/2,0} & \frac{h^2}{4\tau} + \frac{\nu}{2} & 0 & 0 \\ 0 & 0 & \frac{h^2}{4\tau} + \frac{\nu}{2} & -\frac{\nu}{2} - \frac{h}{4}b_{i+1/2,0} \\ 0 & 0 & -\frac{\nu}{2} - \frac{h}{4}b_{i+1/2,0} & \frac{h^2}{4\tau} + \frac{\nu}{2} \end{bmatrix}$$

$$\cdot [v_{i,-1/2}, \; v_{i,1/2}, \; v_{i+1,-1/2}, \; v_{i+1,1/2}]^T.$$

But this time values $v_{i,\pm 1/2}$ and $v_{i+1,\pm 1/2}$ are not independent because of (3.127) that involves

$$v_{i,-1/2} = -v_{i,1/2} \;\text{ and }\; v_{i+1,-1/2} = -v_{i+1,1/2}.$$

These equalities transform elemental matrix:

$$[w^h_{i,-1/2}, \; w^h_{i,1/2}, \; w^h_{i+1,-1/2}, \; w^h_{i+1,1/2}]$$

$$
\cdot \begin{bmatrix} -\dfrac{h^2}{4\tau} - \nu - \dfrac{h}{4} b_{i+1/2,0} & 0 \\ \dfrac{h^2}{4\tau} + \nu - \dfrac{h}{4} b_{i+1/2,0} & 0 \\ 0 & -\dfrac{h^2}{4\tau} - \nu - \dfrac{h}{4} b_{i+1/2,0} \\ 0 & \dfrac{h^2}{4\tau} + \nu - \dfrac{h}{4} b_{i+1/2,0} \end{bmatrix} \begin{bmatrix} v_{i,1/2} \\ v_{i+1,1/2} \end{bmatrix}.
$$

One can see that implementation of this elemental matrix will give equations for $z_{i,1/2}$ and $z_{i+1,1/2}$ which are not uniform with other $z_{i,j+1/2}$ inside domain.

The situation is not better if we use equality from (3.125) to exclude fictitious values

$$
w^h_{i,-1/2} = -w^h_{i,1/2} + 2g^{k+1}_{1,i,0} \quad \text{and} \quad w^h_{i+1,-1/2} = -w^h_{i+1,1/2} + 2g^{k+1}_{1,i+1,0}.
$$

Indeed elemental matrix becames simpler,

$$
[w^h_{1,1/2},\ w^h_{i+1,1/2}] \begin{bmatrix} \dfrac{h^2}{2\tau} + 2\nu & 0 \\ 0 & \dfrac{h^2}{2\tau} + 2\nu \end{bmatrix} \begin{bmatrix} v_{i,1/2} \\ v_{i+1,1/2} \end{bmatrix},
$$

with corresponding addition into the right-hand side. But again it disturbs the uniformity of equations.

Thus, the approach with small fictitious domains does not give in this situation uniformity of algebraic equations near the boundary Γ_y and becomes usefulness in this subproblem for practical use.

But another idea is fruitful for approach with integration over Ω. Under detail consideration we can see that unknowns $w^h_{i,j+1/2}$ at boundary Γ_x can be omitted in further algorithmic considerations by use of boundary conditions, for example,

$$
a_{i,j+1/2} = g^k_{1,i,j+1/2},\ b^*_{i,j} = g^k_{2,i,j},\ \text{etc} \tag{3.128}
$$

at the boundary nodes. It allows us to solve system of equations (3.32), (3.49), (3.53) and exclude 6 different types of equations (3.33), (3.34), (3.50)–(3.52), (3.54). System of algebraic equations becomes more uniform and we shall use this simplification in our numerical experiment.

Another situation is in the subproblem (3.4)–(3.10). Take extended domain Ω_1 and prolong the equation (3.8) by smooth way into additional strips $\Omega_1 \setminus \Omega$ through boundary Γ_y using Taylor expansions of functions in left-hand side of (3.8). Recompute right-hand side of (3.8) with the help of extended functions in fictitious domains, we get equation (3.63) to be valid in extended domain Ω_1. Similarly by Taylor expansions we prolong boundary functions g_1^{k+1} on 4 segments $\{0,1\}\times(-h/2,0)$ and $\{0,1\}\times(1,1+h/2)$. It gives the boundary condition

$$u = g_1^{k+1} \text{ on extended segments } \{0,1\}\times(-h/2, 1+h/2). \tag{3.129}$$

With these extensions we obtain the following Galerkin formulation instead of (3.68): *find* $u^h \in H_x$ *which satisfies the boundary condition*

$$u^h_{i,j+1/2} = g^{k+1}_{1,i,j+1/2}, \quad i = 0, n,\ j = -1, \ldots, n \tag{3.130}$$

and the integral relation

$$\begin{aligned} &\frac{1}{\tau}(u^h, v)_{\Omega_1} + \nu\left(\frac{\partial u^h}{\partial x}, \frac{\partial v}{\partial x}\right)_{\Omega_1} + \frac{1}{2}\left(a\frac{\partial u^h}{\partial x}, v\right)_{\Omega_1} \\ &-\frac{1}{2}\left(au^h, \frac{\partial v}{\partial x}\right)_{\Omega_1} = \frac{1}{\tau}(u^h, v)_{\Omega_1} + \frac{1}{4}\left(f_1^{k+1}, v\right)_{\Omega_1} \end{aligned} \tag{3.131}$$

for an arbitrary function $v \in H_y$ *which satisfies the boundary condition*

$$v_{i,j+1/2} = 0, \quad i = 0, n, \quad j = -1, \ldots, n. \tag{3.132}$$

Repeating considerations like (3.70)–(3.85) on extended domain Ω_1 we get the same equations like (3.72) for $j = 0, \ldots, n-1$ and some equations for $j = -1$ and $j = n$. Last equations later does not influence on approximate solution and we shall omit them in out further algorithmic considerations. Thus, we have the system of algebraic equations

$$\begin{aligned} &\left(-\nu - \frac{h}{2}a^*_{i-1/2,j+1/2}\right)u^h_{i-1,j+1/2} + \left(\frac{h^2}{\tau} + 2\nu\right)u^h_{i,j+1/2} \\ &+ \left(-\nu + \frac{h}{2}a^*_{i+1/2,j+1/2}\right)u^h_{i+1,j+1/2} = \frac{h^2}{\tau}w^h_{i,j+1/2} + \frac{h^2}{4}f^{k+1}_{1,i,j+1/2}, \\ &\qquad i = 1, \ldots, n-1;\ j = 0, \ldots, n-1. \end{aligned} \tag{3.133}$$

To close the system we add the boundary conditions

$$u^h_{i,j+1/2} = g^{k+1}_{1,i,j+1/2}, \quad i = 0, n, \quad j = 0, \ldots, n-1. \tag{3.134}$$

By usual way we demonstrated before it is proved that obtained system is stable upon the initial data and right-hand side in norm $\|\cdot\|_{1,h}$.

So, for subproblem (3.8)–(3.10) the approach with small fictitious domains produces uniform system of algebraic equations that is useful for coding and will be used in our numerical experiments.

Situation with subproblem (3.89)–(3.91) is the same like in (3.6)–(3.7), i.e., approach with small fictitious domains does not give more uniform equations. But second idea is fruitful when unknowns and equations at boundary nodes $z_{i+1/2,0}$ and $z_{i+1/2,n}$ are omitted. As a result we get the system of algebraic equations (3.100), (3.103), (3.108). This trick allows to exclude 6 more types of equations (3.101), (3.102), (3.105)–(3.107), (3.109) and makes simpler coding. Again we shall use this idea in our numerical experiments.

And vica versa situation with subproblem (3.92)–(3.94) is similar to subproblem (3.8)–(3.10). Therefore take extended domain $\Omega_2 = (-h/2,\ 1+h/2) \times (0,\ 1)$ and prolong equation (3.92) into additional strips $\Omega_2 \setminus \Omega$ through Γ_x by Taylor expansions of functions in its left-hand side. Recompute right-hand side of (3.92) with the help of extended functions into fictitious domain and get (3.92) to be valid on extended domain Ω_2. Similarly by Taylor expansions we prolong boundary function g_2^{k+1} on four segments $(-h/2, 0) \times \{0, 1\}$ and $(1, 1+h/2) \times \{0, 1\}$. It gives the boundary condition

$$\bar{w} = g_2^{k+1} \ \text{on} \ \text{ extended } \ \text{segments}\ (-h/2, 1+h/2)\ \times\ \{0,1\}. \tag{3.135}$$

With these extensions we obtain the following Galerkin formulation instead of (3.113)–(3.115): *find* $\bar{w}^h \in H_y$ *which satisfies the boundary condition*

$$\bar{w}^h_{i,j} = g^{k+1}_{2,i,j}, \quad i = 0, n,\ j = 0, \ldots, n, \tag{3.136}$$

and the integral relation

$$\begin{aligned} &\frac{1}{\tau}(\bar{w}^h, v)_{\Omega_2} + \nu \left(\frac{\partial \bar{w}^h}{\partial y},\ \frac{\partial v}{\partial y}\right)_{\Omega_2} + \frac{1}{2}\left(\frac{\partial \bar{w}^h}{\partial y}, v\right)_{\Omega_2} \\ &-\frac{1}{2}\left(b\bar{w}^h, \frac{\partial v}{\partial y}\right)_{\Omega_2} = \frac{1}{\tau}(\bar{u}^h, v)_{\Omega_2} + \frac{1}{4}\left(f_2^{k+1}, v\right)_{\Omega_2} \end{aligned} \tag{3.137}$$

for an arbitrary function $v \in H_y$ *which satisfies the boundary condition*

$$v_{i,j} = 0, \quad i = 0, n, \quad j = 0, \ldots, n. \tag{3.138}$$

Considerations like (3.70)–(3.82) on extended domain Ω_2 give the same equations like (3.133) for $\bar{w}^h_{i+1/2,j}$, $i = 0, \ldots, n-1$, and some equations for $\bar{w}^h_{-1/2,j}$ and $\bar{w}^h_{n+1/2,j}$. Last equations later does not influence on approximate solution and we shall omit them in our further algorithmic considerations. Thus, we have the system of algebraic equations

$$\left(-\nu - \frac{h}{2} b^*_{i+1/2,j-1/2}\right) \bar{w}^h_{i+1/2,j-1} + \left(\frac{h^2}{\tau} + 2\nu\right) \bar{w}^h_{i+1/2,j}$$

$$+ \left(-\nu + \frac{h}{2} b^*_{i+1/2,j+1/2}\right) \bar{w}^h_{i+1/2,j+1} = \frac{h^2}{\tau} \bar{u}^h_{i+1/2,j} + \frac{h^2}{4} f^{k+1}_{2,i+1/2,j}, \qquad (3.139)$$

$$i = 0, \dots, n-1; \ j = 1, \dots, n-1,$$

with boundary condition

$$\bar{w}^h_{i+1/2,j} = g^{k+1}_{2,i+1/2,j}, \quad i = 0, \dots, n-1, \quad j = 0, n. \qquad (3.140)$$

By usual way we demonstrated before, it is proved that obtained system is stable upon the initial data and right-hand side in norm $\|\cdot\|_{2,h}$.

Thus, for the subproblem (3.92)–(3.94) the approach with small fictitious domains produces the uniform system of linear algebraic equations that is useful for coding and will be used in our numerical experiments.

Numerical experiments with selected approach give negative and positive experience which we discuss in the end of paper. We shall come to some recommendations which we shall follow in our further work. In spite of official completion of joint project the laied scientific cooperation and series of theoretical ellaborations will be realized in the form of joint preprints, programs, and papers. So that recommendations of this paper and their realizations will be theoretically and numerically justified and successively published in joint preprints.

4 Numerical experiment

For numerical experiment we take the problem (1.1)–(1.4) with the parameters $\nu = 0.01$, $T = 1$, and the following data:

$$\begin{aligned}
&\left.\begin{aligned} f_1(x,y,t) &= 0, \\ f_2(x,y,t) &= 0, \end{aligned}\right\} && \text{on} \quad \Omega \times (0,T); \\
&\left.\begin{aligned} g_1(x,y,t) &= -\cos(\pi x)\sin(\pi y)\exp(-2\pi^2 \nu t), \\ g_2(x,y,t) &= \quad \sin(\pi x)\cos(\pi y)\exp(-2\pi^2 \nu t), \end{aligned}\right. && \text{on} \quad \Gamma \times (0,T); \\
&\left.\begin{aligned} u_{0,1}(x,y) &= -\cos(\pi x)\sin(\pi y), \\ u_{0,2}(x,y) &= \quad \sin(\pi x)\cos(\pi y). \end{aligned}\right. && \text{on} \quad \Omega.
\end{aligned}$$

The solution of this problem is

$$
\begin{aligned}
u_1(x,y,t) &= -\cos(\pi x)\sin(\pi y)\exp(-2\pi^2\nu t),\\
u_2(x,y,t) &= \sin(\pi x)\cos(\pi y)\exp(-2\pi^2\nu t),\\
p(x,y,t) &= -0.25\Big(\cos(2\pi x)+\cos(2\pi y)\Big)\exp(-4\pi^2\nu t),
\end{aligned}
$$

on $\bar{\Omega}\times[0,T]$. The graphs of these functions are presented on Fig. 7, 9, 11, at time $t=1$. We solve the discrete problem using small fictitious domain approach from 3.3 to simplify a coding.

First we consider the error for pressure. The Fig. 15 demonstrates that the order of convergence is $\tau^{1/2}+h^2$ in discrete L_2-norm. On Fig. 12 we see the artificial numerical boundary layer specially in corners, which is usual for splitting. Its origin comes from incorrect boundary conditions for pressure of Neumann type. For example, at the point $(x,0)\in\Gamma$ from initial equation (1.7) we have

$$
\frac{\partial p}{\partial n}=f_2-\frac{\partial u_2}{\partial t}+\nu\Delta u_2-\frac{1}{2}(\mathbf{u}\cdot\nabla)u_2-\frac{1}{2}div(u_2\mathbf{u}).
$$

And on the fractional step of pressure work we get on the base of (1.12) and previous considerations that for splitting problem we have an equality equivalent to

$$
\frac{\partial p}{\partial n}=f_2-\frac{\partial u_2}{\partial t}.
$$

So you see that we have the error of order $O(1)$ in boundary condition. It is a good luck that this discrepancy produces the error in pressure only in narrow boundary layer, which gives small error in L_2-norm and in its discrete analogue.

Fig. 13, 14 demonstrate the order of convergence $\tau+h^2$ for both components of velocities in discrete L_2-norm. The error for velocities also has an artificial numerical boundary layers that is demonstrated by Fig. 8, 10. Here origin of artificial boundary layers comes from splitting into geometrical directions and is produced by unsimultaneous use of boundary conditions. For example, in problem (3.6)–(3.7) for u_1 we first use boundary condition like $u_1=g_1$ on $\Gamma_y\times[t_k,t_{k+1}]$. It means that u_1 on fractional step in y-direction in general does not satisfy boundary condition $u_1=g_1$ on $\Gamma_x\times[t_k,t_{k+1}]$ with discrepancy of order τ. Therefore on the next fractional step in k-direction (equations (3.8)–(3.10)) we have

$$
\lim_{x\to 0}\frac{\partial u}{\partial t}(x,y)=\frac{\partial u_1}{\partial t}(0,y)+O(1).
$$

Again it produces approximation error of order $O(1)$ in thin vicinity of Γ_x that results in artificial boundary layer of amplitude $O(\tau)$.

Analogously we get thin artificial boundary layer for u_2 in the vicinity of Γ_y. Of course, these boundary layers with small amplitude give the accuracy corresponding truncation error. But they do not give to use Richardson extrapolation for increase of accuracy order because of irregular character of approximate solution error.

5 Conclusions

5.1 Splitting method vs. solution with complete operator

An important advantage of splitting method is reduction of problem with complete operator into several simpler problems on each time step: Poisson-like problem for pressure and four families of one-dimensional problems (in view of discretization in time). Operator of Poisson-like problem is symmetric and positive-definite, has constant coefficients. It allows to use many effective algorithms to solve the problem. It results in algebraic complexity with number N of arithmetical operations, where $N \approx cn^2$ is number of unknowns.

The main disadvantage of splitting method consists of artificial boundary layers produced by inaccurate boundary conditions. As it was written yet, they have comparatively small amplitude but have irregular character and do not give to increase the accuracy by Richardson extrapolation. Of course, there are several papers (for example, see [16], [24] and references in it) in which amplitude of artificial boundary layers is somewhat reduced because of more accurate work with boundary conditions. Another ways to get the second order of convergence in time consist of Crank-Nicholson approach and Θ-scheme [12], [8], [13].

But in principle, Richardson extrapolation for regular truncation error and stable scheme allows any finite order of convergence, for example, third and fourth. Such a regular truncation error is given by full implicit scheme.

Therefore on next stage of our joint work we shall use full implicit scheme to ensure an increase of convergence order at least in τ.

5.2 Staggerred meshes vs. united mesh

The main advantage of staggerred meshes consists of automatic fulfilment of LBB-condition for pressure stability [2]. But last years the other approach is popular enough: filtering the spurious modes. The main idea is to implement united square mesh and bilinear finite element for velocities and piecewise constant for pressure. This scheme becomes stable with orthogonalization of approximate solution to local spurious modes [1], [2], [14]. In principle, this orthogonalization reduces the number of degrees of freedom for pressure from n^2 to $3/4\ n^2$ in $2D$-problem. For $3D$-problem this loss is even less: $7/8\ n^3$ instead of n^3 [14]. Algebraic complexity due to orthogonalization increases by $2n^2$ arithmetical operations only.

But advantage of united mesh is evident. The coding for united mesh is simpler even in $2D$-problem. In the vicinity of curvilinear boundary this reason becomes crucial since staggerred meshes come to multiciphered approximations of domain that is problematic from both theoretical and practical points of view.

Therefore on the next stage of our joint work we shall use united mesh with filtering of local spurious model instead of staggerred meshes.

5.3 Square vs. triangle mesh

It looks that square mesh is more appropriate for our problem. First, in $2D$ domain the number of squares is twice less than the number of triangles. For $3D$ domain this ratio is usually between 5 and 6. It produces the greater job with simplex elements. Then, the quadrature formulae are simpler for square than for triangle that is more considerable in $3D$ elements. But triangles give the better possibilities to approximate a curvilinear boundary.

Therefore we shall use at next stage the combination of square mesh in the domain with triangle elements in the thin vicinity of curvilinear boundary. Of course, in the situation with condenced meshes in adaptive approach we get some "nonconforming approach" from the elemental point of view. But from nodal point of view this approach with dividing square in m^2 equal squares is conforming and has no difficulties in theoretical justification and practical assembling.

We shall not accumulate further results in the form of (third) volume as it was in two previous cases. To accelerate exchange of results we shall publish our materials in form or preprints and papers as far as they will appear.

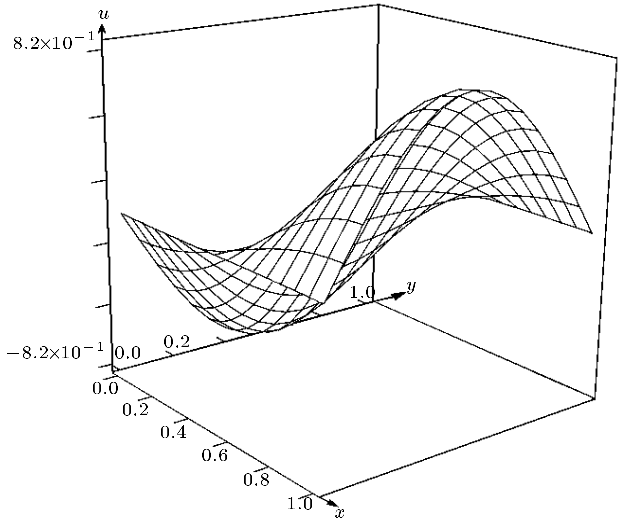

Fig. 7. The first component of velocities values.

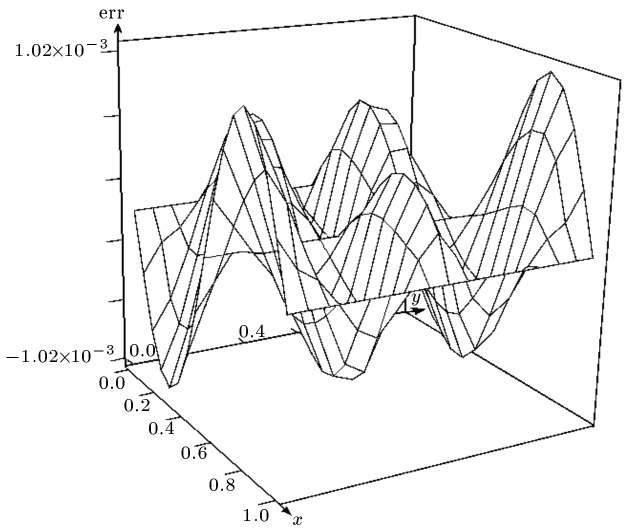

Fig. 8. Errors for the first component of velocity.

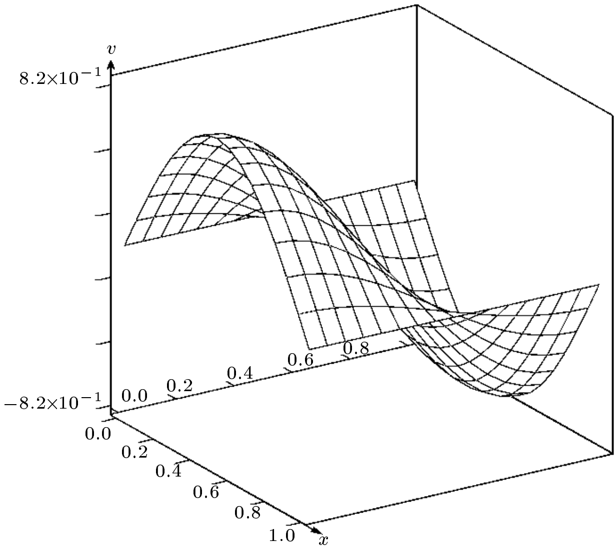

Fig. 9. The second component of velocities values.

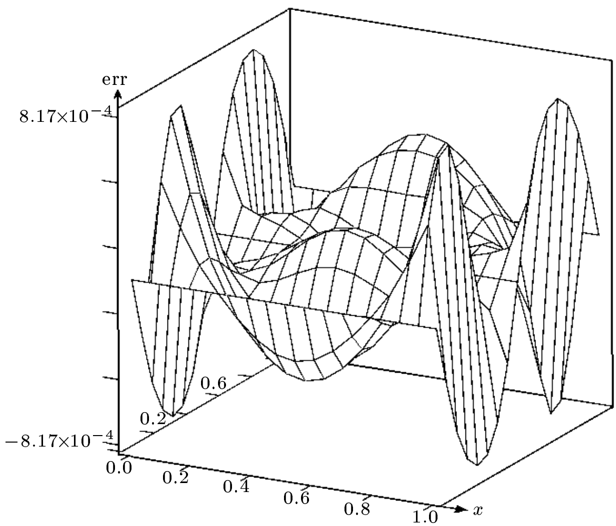

Fig. 10. Errors for the second component of velocity.

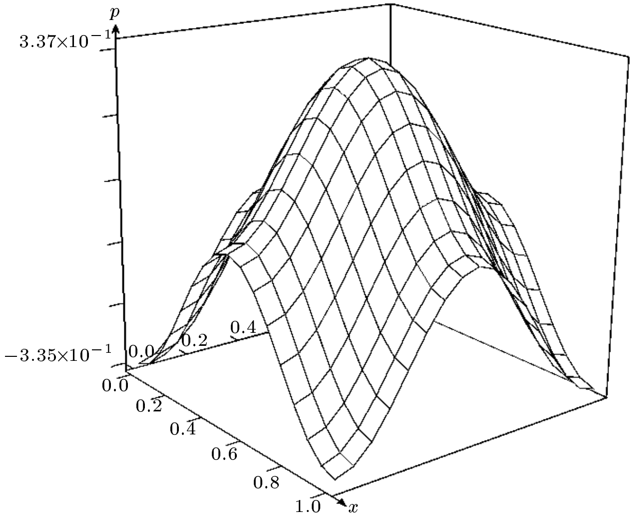

Fig. 11. The pressure values.

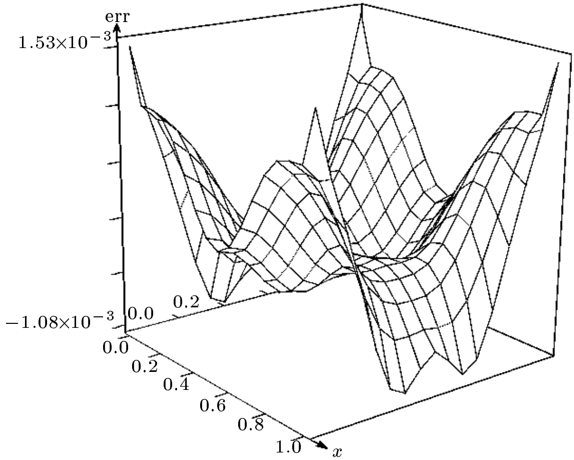

Fig. 12. Errors for the pressure.

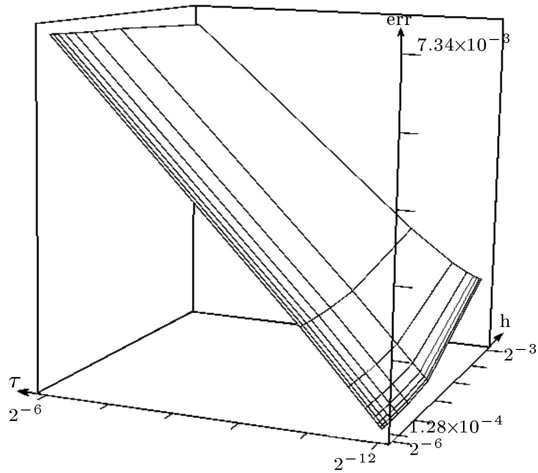

Fig. 13. The dependence of maximal on $t \in [0,\ 1]$ $L_2(\Omega)-$norm of $u-$error on $h,\ \tau$.

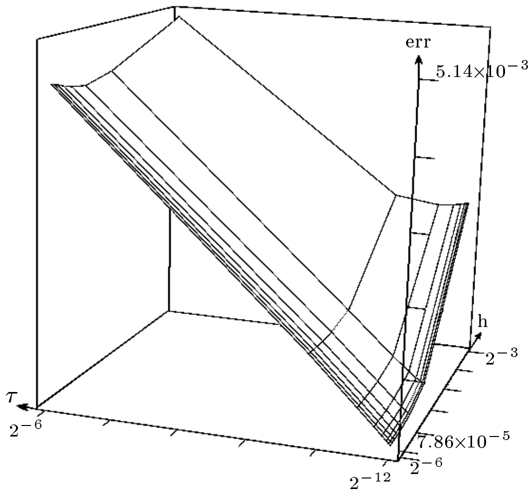

Fig. 14. The dependence of maximal on $t \in [0,\ 1]$ $L_2(\Omega)-$norm of $v-$error on $h,\ \tau$.

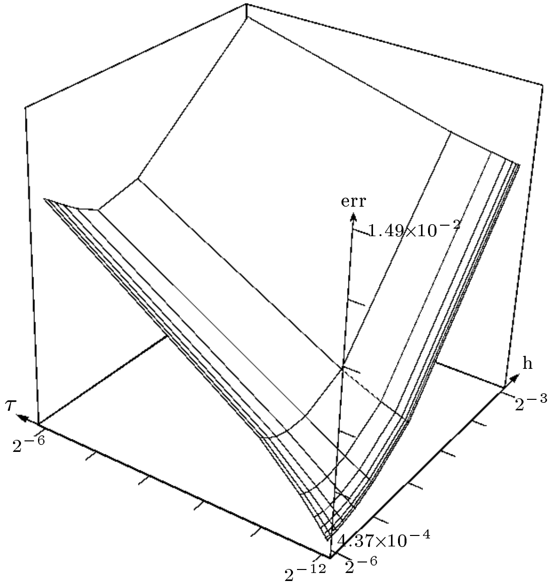

Fig. 15. The dependence of maximal on $t \in [0,\ 1]$ $L_2(\Omega)$−norm of pressure errors on h, τ.

References

1. Boland J.M., Nicolaides R.A. *Stability of finite elements under divergence constants.* - SIAM J. Numer. Math., 1983, v.20, p. 722-731.
2. Brezzi F., Fortin M. *Mixed and Hybrid Finite Element Metods.* New York: Springer, 1991.
3. Chen H., Rannacher R. *Superconvergence properties of finite element schemes for the Navier-Stokes problem.* Preprint 93-37, SFB 359, Universität Heidelberg, 1993.
4. Chorin A.J. *A numerical method for solving incompressible viscous flow problems.* J. of Comp. Phys., 1967, v.2, p. 12-26.
5. Chorin A.J. *Numerical solution of the Navier-Stokes equations.* Math. Comp., 1968, v.2, p. 745-762.
6. Chorin A.J. *Numerical solution of incompressible flow problems.* Studies in Num. Anal., 1968, v.2, p. 64-71.
7. Chorin A.J. *On the convergence of discrete approximations of the Navier-Stokes equations.* Math. Comp., 1969, v.23, p. 341-353.
8. Glowinski R. *Le Θ−scheme.* In: Numerical methods for Navier-Stokes equations, Bristian M.O., Glowinski R., Perinx J., eds., p.73-87, 1987.
9. Hackbusch W. *Multigrid Methods and Applications.* Berlin: Springer, 1985.

10. Heywood J.G., Rannacher R. *Finite element approximation of the nonstationary Navier-Stokes Problem. I. Regularity of solutions and second order error estimates for spatial discretization.* - SIAM J. Numer. Anal., 1982, v.19, p. 275-311.
11. Heywood J.G., Rannacher R. *Finite element approximation of the nonstationary Navier-Stokes Problem. II. Stability of solutions and error estimates uniform in time.* - SIAM J. Numer. Anal., 1986, v.23, p. 750-777.
12. Heywood J.G., Rannacher R. *Finite element approximation of the nonstationary Navier-Stokes Problem. IV. Error analysis for second-order time discretization.* - SIAM J. Numer. Anal., 1990, v.17, p. 353-384.
13. Kloucek P., Rys F. *Stability of the Fractional $\Theta-$ scheme for Nonstationary Navier-Stokes Equations.* - SIAM J. Numer. Anal., 1994, v.31, p. 1312-1336.
14. Mansfield L. *On Finite Element Subspaces over Quadrilateral and Hexahedral Meshes for Incompressible Viscous Flow Problems.* - Mumer. Math., 1984, v.45, p. 165-172.
15. Marchuk G.I., Shaidurov V.V. *Difference Methods and Their Extrapolations.* New York: Springer, 1983.
16. Prohl A. *Projection and Quasi-Compressibility Methods for Solving the Incompressible Navier-Stokes Equations.* Studtgart: Teubrer, 1997.
17. Rannacher R. *Finite Element Methods for the Incompressible Navier-Stokes Equations.* In: Fundamental Directions in Mathematical Fluid Mechanics. Galdi G., Heywood J.G., Rannacher R., eds. Berlin: Birkhäuser, 2000.
18. Rüde U. *Multilevel, extrapolation, and sparse grid methods.* In Multigrid Methods IV: Proceedings of the Fourth European Conference on Multigrid Methods, p. 281–294, Boston, 1994.
19. Stynes M., Roos H.-G., Tobiska L. *Numerical methods for sigularly perturbed differential equations. Convection-diffusion and flow problems.* Berlin: Springer, 1996.
20. Tirek S. *Efficient Solvers for Incompressible Flow Problems.* Berlin - Heidelberg: Springer, 1999.
21. Shaidurov V.V. *Multigrid Methods for Finite Elements.* Dordrecht: Kluwer Academic Publishers, 1995.
22. Shen J. *On error estimates of the projection metods for the Navier-Stokes Equations: second-order schemes.* - Math. Comp., 1996, v. 65, № 215, p. 1039-1065.
23. Temam R. *Navier-Stokes Equations. Theory and Numerical Analysis.* Amsterdam: North-Holland Publishing Company, 1979.
24. Van Kan. *A second-order accurate pressure-correction scheme for viscous incompressible flow.* - SIAM J. Sci. Staf. Comp., 1986, v.7, p. 870-891.