A difference scheme for convection-diffusion problem on the oriented grid

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Introduction

The work is devoted to a difference method for solving two-dimensional problem for convection-dominated convection-diffusion equation. This problem is related to the class of singular disturbed problems and it often has a solution of a boundary layer type with strong increase of derivatives in a vicinity of certain lines and points [1, 2, 3].

An application of the finite element method or difference methods for such problems has some specific features in comparison with the boundary value problem when convection and diffusion items have the same order. First, in zone of boundary layer it is necessary to take into consideration the boundary layer type of the solution [2] or to condense grid to compensate strong increase of derivatives [3]. Second, in zone of smoothness, when the influence of higher derivatives is low, we should take into account that the equation becomes the convection one (called here as reduced equation), while the area of solution dependence in points of this zone tends to a piece of reduced equation characterictic. Third, the standard difference schemes and the schemes of the finite element method with central differences lose a stability, while the schemes with directional differences possess computational diffusion which is essentially greater than the physical one and it disturbs even qualitative description of solution, not to mention the quantitative similarity. In contrast to the physical diffusion, the computational one differs both in various space points and in various directions in the same point. The role of the "longitudinal" computational diffusion, i.e., the diffusion along the convective flow, is already evidently seen in one-dimensional case, where it studied well and give the same consequences as in two-dimensional problems. In section 3, we define more precisely the influence of the "transversal" computational diffusion, which "washes-out" the difference solution in nontangent directions to convective flow. In certain difference schemes it essentially exceeds the physical diffusion, therefore to check it, we introduce the value, which is called the criterion of grid orientation along the convective flow.

In section 5, we state the algorithm of successive strengthening orientation for an arbitrary grid without new inner nodes addition and without node coordinates modification. In section 6, this algorithm is illustrated with an example of grids with uniform arrangement of nodes, but more and more oriented along the flow at the expence of changing stencil topology of difference scheme.

In section 4, we suggest the method of construction of inverse-monotone second-order finite-difference scheme. The combination of these properties is usually reached by special matching of the flow direction and the arrangement of grid nodes. Such, for example, is Crank-Nikolson scheme for convective term approximation with the arrangement of two nodes along the flow in the characterictic method. The use of the strengthening orientation algorithm provides this opportunity for arbitrary arrangement of the grid nodes.

1 The difference problem statement

Let us introduce Euclidean distance $|z - z'| = ((x - x')^2 + (y - y')^2)^{1/2}$ between two points $z = (x, y)$ and $z' = (x', y')$ in R^2. Let $\Omega = \{z = (x, y) : 0 < x < 1,\ 0 < y < 1\}$ be opened unit square with boundary Γ.

We shall use notation $C^k(\bar{D})$ in an arbitrary subdomain $D \subset \Omega$ for the class of functions having continuous k-th partial derivatives on closure $\bar{D}$ with the norm

$$\|u\|_{k,\bar{D}} = \max_{\alpha_1+\alpha_2 \le k} \max_{\bar{D}} \left| \frac{\partial^{\alpha_1+\alpha_2} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right|$$

where α_1, α_2 are non-negative integers. Assume that $C^0(\bar{D}) = C(\bar{D})$.

Consider the problem

$$-\varepsilon \Delta u + b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} = f \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = g \quad \text{on} \quad \Gamma, \tag{1.2}$$

where $\varepsilon \ll 1$ is a small positive parameter; functions $b_1, b_2 \in C(\overline{\Omega})$ and the right-hand sides $f \in C(\overline{\Omega}), g \in C(\Gamma)$ are known. Thus, we have a solvable boundary value problem for the elliptic second-order equation [4].

In a subdomain, where the second derivatives are limited, their influence is low due to the small parameter ε. Therefore the equality (1.1) comes to the equation of first order, which characterictic system of ordinary differential equations corresponds

$$\frac{dx}{b_1(x,y)} = \frac{dy}{b_2(x,y)} = \frac{du}{f(x,y)}. \tag{1.3}$$

Its solution is the set of characteristic curves or simply the characteristic. In each points $z = (x, y) \in \overline{\Omega}$ the vector $t(z) = (b_1(z), b_2(z))$ touches the characteristic passing through this point. Therefore, we call it as characteristic vector, while the opposite vector as anticharacteristic one. We assume that a direction is a corresponding vector of unit length. In particular, the direction $(b_1^2 + b_2^2)^{-1/2}(b_1, b_2)$ with $b_1^2 + b_2^2 \neq 0$ in point (x, y) is called as characteristic direction, while any other direction, that does not coincide with it or with the opposite one is called as direction that "transversal" to characteristic one.

2 The difference approximation of convective item on an arbitrary trianqular stencil

Consider the triangle with vertices

$$z_t = (x_t, y_t), \quad z_s = (x_s, y_s), \quad z_r = (x_r, y_r),$$

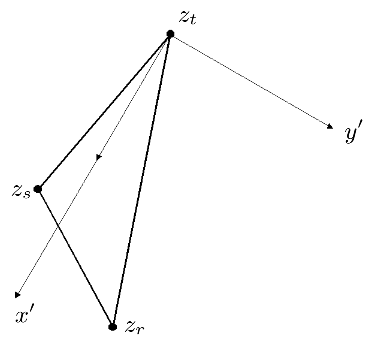

Fig. 1: The triangular stencil and the new local coordinates.

at a distance not greater that h from each other and not lying on one straight line. It is suppose that $b_1^2(z_t) + b_2^2(z_t) \neq 0$ and anticharacteristic vector $-\ t(z_t)$ lies in angle $\angle z_s z_t z_r$ (see Fig. 1).

Let construct the following approximation for this tree-point stencil with the help of indefinite coefficient method [12] :

$$b_1 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} \approx \alpha u(z_t) + \beta u(z_s) + \gamma u(z_r) \tag{2.1}$$

in node z_t. Suppose that u belongs to $C^3(B(z_t, h))$ in the closed ball $B(z_t, h) = \{z : |z - z_t| \leq h\}$.

To simplify the problem, we introduce new local Cartesian coordinates (x', y') with the origin in z_t and with axis Ox' along $-t(z_t)$ (see Fig. 1). The vector $\overline{b} = (b_1, b_2)$ in new coordinates comes to $\overline{b'} = (b'_1, b'_2)$ with coordinates $b'_1 = (b_1^2 + b_2^2)^{1/2}$, $b'_2 = 0$. Points $z_s = (x_s, y_s)$, $z_r = (x_r, y_r)$, $z_t = (x_t, y_t)$ comes to $z'_s = (x'_s, y'_s)$, $z'_r = (x'_r, y'_r)$, $z'_t = (0, 0)$, and function $u(z)$ does in $\tilde{u}(z')$ respectively. The item in the right-hand side (2.1) comes to $(\, b_1^2 \, + \, b_2^2 \,)^{1/2}\, \partial \tilde{u}/\partial x'$. Let take Taylor series with respect to $z'_t = (0, 0)$ for function $\tilde{u}(z')$, summate them and $\tilde{u}(z'_t)$ with indefinite weights α, β, γ :

$$\begin{aligned} &\alpha u(z_t) + \beta u(z_s) + \gamma u(z_r) = (\alpha + \beta + \gamma)\, \tilde{u}(z'_t) \\ &+ \beta x'_s + \gamma x'_r)\, \frac{\partial \tilde{u}}{\partial x}(z'_t) + (\beta y'_s + \gamma y'_r)\, \frac{\partial \tilde{u}}{\partial y}(z'_t) \\ &+ c'_1 \frac{\partial^2 \tilde{u}}{\partial x^2}(z'_t) + c'_2 \frac{\partial^2 \tilde{u}}{\partial x \partial y}(z'_t) + c'_3 \frac{\partial^2 \tilde{u}}{\partial y^2}(z'_t) + O(h^3) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} c'_1 &= \frac{1}{2}\, \beta\, {x'_s}^2 + \frac{1}{2}\, \gamma\, {x'_r}^2, \\ c'_2 &= \beta\, x'_s y'_s + \gamma\, x'_r y'_r, \\ c'_3 &= \frac{1}{2}\, \beta\, {y'_s}^2 + \frac{1}{2}\, \gamma\, {y'_r}^2. \end{aligned} \tag{2.3}$$

Here, we can distinctly see the computational diffusion $c'_1\, \partial^2 \tilde{u}/\partial {x'}^2$ along the characteristic line, diffusion $c'_3\, \partial^2 \tilde{u}/\partial {y'}^2$ in perpendicular direction, and diffusion $c'_2\, \partial^2 \tilde{u}/\partial x' \partial y'$ in some intermediate directions.

Since h is small enough, in order to get at least the first order of approximation, we need the following equalities:

$$\begin{aligned} \alpha + \beta + \gamma &= 0, \\ \beta\, x'_s + \gamma\, x'_r &= b'_1, \\ \beta\, y'_s + \gamma\, y'_r &= 0. \end{aligned} \tag{2.4}$$

The matrix's determinant of this system equals double square of triangle $\triangle z_t z_s z_r$, which is denoted as S. Since the triangle does not degenerate into a line or a point, the determinant is not equal to zero and the system has the unique solution

$$\begin{aligned}\alpha &= (b_2(x_r - x_s) - b_1(y_r - y_s))/S,\\ \beta &= (b_1(y_r - y_t) - b_2(x_r - x_t))/S,\\ \gamma &= (b_2(x_s - x_t) - b_1(y_s - y_t))/S,\end{aligned} \tag{2.5}$$

where $S = (y_r - y_t)(x_s - x_t) - (x_r - x_t)(y_s - y_t) = y'_r x'_s - x'_r y'_s$.

Using β and γ in (2.3), we obtain:

$$\begin{aligned}c'_1 &= \frac{y'_r {x'}_s^2 - y'_s {x'}_r^2}{y'_r x'_s - x'_r y'_s},\\ c'_2 &= \frac{y'_s y'_r}{y'_r x'_s - x'_r y'_s}(x'_s - x'_r),\\ c'_3 &= \frac{y'_s y'_r}{y'_r x'_s - x'_r y'_s}(y'_s - y'_r).\end{aligned} \tag{2.6}$$

Hence, to decrease the coefficients c'_2 and c'_3, we need to minimize the following value

$$Kr(z_t) = y'_s y'_r / S \tag{2.7}$$

which is called *the index of triangle's orientation* $\Delta z_t z_z z_r$ *in point* z_t.

It should be noted that if y'_s or y'_r is zero, then the approximating transversal diffusion equals zero. That is the case, for example, in the method of characteristics.

3 Construction of inverse-monotone second-order finite-difference scheme

To construct the difference scheme, we first introduce the discrete set Ω_h of nodes in Ω and the discrete set Γ_h of nodes on Γ. Assume that $\bar{\Omega}_h = \Omega_h \cup \Gamma_h$. For each node $z \in \Omega_h$ we form the subset N_z of some nearer nodes of $\bar{\Omega}_h$. Denote by h_z the local radius of this subset:

$$h_z = \max_{z' \in N_z} |z - z'| \sim h.$$

Let us take an arbitrary inner node $\bar{z} \in \Omega_h$ and introduce local orthogonal coordinates ξ, η with origin in $\bar{z}$, with axis $O\xi$ along $t(\bar{z})$ and axis $O\eta$

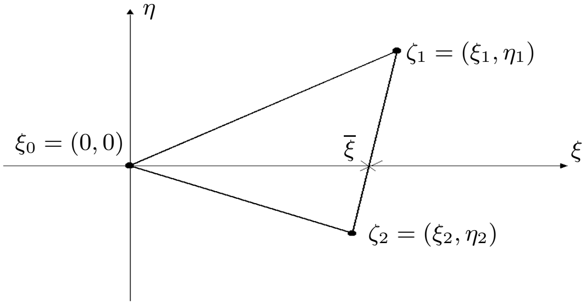

Fig. 2: The local coordinates (ξ, η) and the arrangement of nodes $\zeta_0, \zeta_1, \zeta_2$.

to the left of $t(\bar{z})$ (see Fig.2). In this coordinates equation (2.1) comes to another one:

$$-\varepsilon\tilde{\Delta}\tilde{u} - d\frac{\partial\tilde{u}}{\partial\xi} + \sigma\frac{\partial\tilde{u}}{\partial\eta} = \tilde{f} \tag{3.1}$$

in the h_z-vicinity of node z. Here for any function $w(x, y)$ we put

$$\tilde{w}(\xi, \eta) = w(x(\xi, \eta), y(\xi, \eta)) \tag{3.2}$$

and introduce new functions

$$d(\xi, \eta) = \frac{b_1(\bar{z})\tilde{b}_1(\xi, \eta) + \tilde{b}_2(\bar{z})b_2(\xi, \eta)}{|t(\tilde{z})|},$$

$$\zeta(\xi, \eta) = \frac{b_2(\bar{z})\tilde{b}_1(\xi, \eta) - b_1(\bar{z})\tilde{b}_2(\xi, \eta)}{|t(\tilde{z})|};$$

operator $\tilde{\Delta} = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$ has the same form but in new coordinates.

Further we study two situations separately: $\varepsilon \le c_0^{-2}h_{\bar{z}}^2$ and $c_1^{-2}h_{\bar{z}}^2 < \varepsilon$ with some constants c_0, c_1 independent of $\varepsilon, h_{\bar{z}}$. Let start with the first one.

3.1. Large $h_{\bar{z}}$. Tree-point stencil.

First situation means that

$$h_{\bar{z}} \ge c_0\sqrt{\varepsilon}. \tag{3.3}$$

Suppose that u belongs to $C^3(B(\bar{z}, h_{\bar{z}}))$ in the closed ball $B(\bar{z}, h_{\bar{z}}) = \{z : |z - \bar{z}| \le h_{\bar{z}}\}$ and has bounded norm

$$\|\tilde{u}\|_{3,B(0,h_{\bar{z}})} = \|u\|_{3,B(\bar{z},h_{\bar{z}})} \le c_2 \tag{3.4}$$

with constant c_2 independent of $h_{\bar{z}}$ and ε.

Our goal is to derive an equality

$$\alpha_0\tilde{u}(\zeta_0)+\alpha_1\tilde{u}(\zeta_1)+\alpha_2\tilde{u}(\zeta_2)=\tilde{f}(\zeta_0)+\beta_1\frac{\partial\tilde{f}}{\partial\xi}(\zeta_0)+\beta_2\frac{\partial\tilde{f}}{\partial\eta}(\zeta_0)+O(h_{\bar{z}}^2). \quad (3.5)$$

We consider special arrangement of nodes. It is supposed that $\zeta_0=(0,0)$; node ζ_1 lies in first quadrant: $\xi_1>0$, $\eta_1\geq 0$; and node ζ_2 lies in fourth one: $\xi_2>0$, $\eta_2\leq 0$. Let us take Taylor series in nodes ζ_1,ζ_2 with respect to ζ_0 for function $\tilde{u}$, $\tilde{f}$:

$$\begin{aligned}\tilde{u}(\zeta_i)=\tilde{u}(\zeta_0)+\xi_i\frac{\partial\tilde{u}}{\partial\xi}(\zeta_0)+\eta\frac{\partial\tilde{u}}{\partial\eta}(\zeta_0)+\frac{\xi_i^2}{2}\frac{\partial^2\tilde{u}}{\partial\xi^2}(\zeta_0)\\+\xi_i\eta_i\frac{\partial^2\tilde{u}}{\partial\xi\partial\eta}(\zeta_0)+\frac{\eta_i^2}{2}\frac{\partial^2\tilde{u}}{\partial\eta^2}(\zeta_0)+O(h_{\bar{z}}^3)\end{aligned} \quad (3.6)$$

and

$$\tilde{f}(\zeta_0)=-d(\zeta_0)\frac{\partial\tilde{u}}{\partial\xi}(\zeta_0)+O(h_z^2), \quad (3.7)$$

$$\frac{\partial\tilde{f}}{\partial\xi}(\zeta_0)=-\frac{\partial d}{\partial\xi}(\zeta_0)\frac{\partial\tilde{u}}{\partial\xi}(\zeta_0)-d(\zeta_0)\frac{\partial^2\tilde{u}}{\partial\xi^2}(\zeta_0)+\frac{\partial^2\sigma}{\partial\xi}(\zeta_0)\frac{\partial\tilde{u}}{\partial\eta}(\zeta_0)+O(h_{\bar{z}}), \quad (3.8)$$

$$\frac{\partial\tilde{f}}{\partial\eta}(\zeta_0)=-\frac{\partial d}{\partial\eta}(\zeta_0)\frac{\partial\tilde{u}}{\partial\xi}(\zeta_0)-d(\zeta_0)\frac{\partial^2\tilde{u}}{\partial\xi\partial\eta}(\zeta_0)+\frac{\partial^2\sigma}{\partial\eta}(\zeta_0)\frac{\partial\tilde{u}}{\partial\eta}(\zeta_0)+O(h_{\bar{z}}). \quad (3.9)$$

Now use these decompositions in both sides of (3.5). In order to get at least first order of approximation, we need to cancel terms $\tilde{u},\partial\tilde{u}/\partial\xi,\partial\tilde{u}/\partial\eta$:

$$\alpha_0+\alpha_1+\alpha_2=0, \quad (3.10)$$

$$\alpha_1\xi_1+\alpha_2\xi_2=-d(\zeta_0)-\beta_1\frac{\partial d}{\partial\xi}(\zeta_0)-\beta_2\frac{\partial d}{\partial\eta}(\zeta_0), \quad (3.11)$$

$$\alpha_1\eta_1+\alpha_2\eta_2=\beta_1\frac{\partial\sigma}{\partial\xi}(\zeta_0)+\beta_2\frac{\partial\sigma}{\partial\eta}(\zeta_0). \quad (3.12)$$

Two more equalities follow from elimination of $\partial^2\tilde{u}/\partial\xi^2$, $\partial^2\tilde{u}/\partial\xi\partial\eta$:

$$\frac{1}{2}(\alpha_1\xi_1^2+\alpha_2\xi_2^2)=-d(\zeta_0)\beta_1, \quad (3.13)$$

$$\alpha_1\xi_1\eta_1+\alpha_2\xi_2\eta_2=-d(\zeta_0)\beta_2. \quad (3.14)$$

In principle, we get 5 linear equations for 5 unknowns. Later we shall see that $|\beta_1|,|\beta_2|$ are small enough and

$$\tilde{\zeta}=(\beta_1,\beta_2)\in\Delta\zeta_0\zeta_2\zeta_1. \quad (3.15)$$

It gives us a possibility to change the right-hand side in (3.11) and $-d(\zeta_0)$ in (3.13), (3.14) by $-d(\tilde{\zeta})$ without violation of the second order of approximation:

$$\alpha_1\xi_1 + \alpha_2\xi_2 = -d(\tilde{\zeta}), \tag{3.16}$$

$$\frac{1}{2}(\alpha_1\xi_1^2 + \alpha_2\xi_2^2) = -d(\tilde{\zeta})\beta_1, \tag{3.17}$$

$$\alpha_1\xi_1\eta_1 + \alpha_2\xi_2\eta_2 = -d(\tilde{\zeta})\beta_2. \tag{3.18}$$

Since $\sigma(\zeta_0) = 0$, the same modification may be done in (3.12) without violation of the second order of approximation:

$$\alpha_1\eta_1 + \alpha_2\eta_2 = \sigma(\tilde{\zeta}). \tag{3.19}$$

Equalities (3.10), (3.16), (3.19) give the system with respect to α_i with unique solution

$$\alpha_0 = ((\eta_1 - \eta_2)d(\tilde{\zeta}) + (\xi_1 - \xi_2)\sigma(\tilde{\zeta}))/(2s_{21}),$$

$$\alpha_1 = (\eta_2 d(\tilde{\zeta}) + \xi_2\sigma(\tilde{\zeta}))/(2s_{21}), \tag{3.20}$$

$$\alpha_2 = (\eta_1 d(\tilde{\zeta}) - \xi_1\sigma(\tilde{\zeta}))/(2s_{21}), \tag{3.21}$$

where $s_{21} = (\xi_2\eta_1 - \xi_1\eta_2)/2$ is the area of triangle $\Delta\zeta_0\zeta_2\zeta_1$. Due to (3.15) and equality $\sigma(\zeta_0) = 0$, the inequalities

$$|\sigma(\tilde{\zeta})| \le ch_{\bar{z}} \ll d(\tilde{\zeta}) \tag{3.22}$$

hold. Therefore when η_1 is comparable with ξ_1, i.e., $\eta_1 \sim \xi_1$, we get

$$\alpha_2 \le 0; \tag{3.23}$$

analogously from comparability of $|\eta_2|$ with ξ_2 it follows that

$$\alpha_1 \le 0. \tag{3.24}$$

Both previous inequalities involve

$$\alpha_0 \le 0. \tag{3.25}$$

It would give M-property of the difference operator in the left-hand side of (4.5).

Now let us use (3.20) in (3.17) and (3.18):

$$\beta_1 = \frac{1}{4s_{21}}((\eta_1\xi_2^2 - \eta_2\xi_1^2) + \xi_1\xi_2(\xi_2 - \xi_1)\sigma(\tilde{\zeta})/d(\tilde{\zeta})), \tag{3.26}$$

$$\beta_2 = \frac{1}{2s_{21}}((\xi_2 - \xi_1)\eta_1\eta_2 + \xi_1\xi_2(\eta_2 - \eta_1)\sigma(\tilde{\zeta})/d(\tilde{\zeta})). \tag{3.27}$$

From arrangement of ζ_i it follows that

$$0 \le \beta_1 \le \frac{1}{2}\max\{\xi_1, \xi_2\} \le h_{\bar{z}}/2, \tag{3.28}$$

$$\eta_2 \le \beta_2 \le \eta_1, \ |\beta_2| \le h_{\bar{z}}. \tag{3.29}$$

So, you see that β_1, β_2 are small enough and was found by unique way with (3.26), (3.27). After that one can find α_i from (3.20) with the help of equality

$$d(\tilde{\zeta}) = d(\beta_1, \beta_2).$$

Finally, in order to get second order of approximation we need coefficient A_{22} before $\partial^2\tilde{u}/\partial\eta^2$ in (3.6) to be small enough:

$$|A_{22}| = \left|\alpha_1\frac{\eta_1^2}{2} + \alpha_2\frac{\eta_2^2}{2}\right| \le c_3 h_{\bar{z}}^2. \tag{3.30}$$

Since A_{22} is positive, we need only

$$A_{22} = -\eta_1\eta_2(\eta_1 - \eta_2)d(\tilde{\zeta})/(4s_{21}) \le c_3 h_{\bar{z}}^2. \tag{3.31}$$

Let $\tilde{\zeta}$ is cross-point of edge ζ_1, ζ_2 with axis $O\xi$, then

$$s_{21} = \frac{1}{2}(\eta_1 - \eta_2)\tilde{\xi}. \tag{3.32}$$

Combining it with (3.31) we get

$$-\frac{\eta_1\eta_2}{2\tilde{\xi}}d(\tilde{\zeta}) \le c_3 h_{\bar{z}}^2. \tag{3.33}$$

In principle, $\tilde{\xi} \sim h_{\bar{z}}$. Therefore we need

$$-\eta_1\eta_2 \sim c_4 h_{\bar{z}}^3. \tag{3.34}$$

From the first sight it seems to be unusual since the left-hand side has only second order of smallness. But in the next section we shall describe an algorithm of grid reorientation which gives this inequality and (3.31) by regular way. Therefore we consider inequality (3.31) to be valid.

3.2. Small $h_{\bar{z}}$. Five-point stencil.

Second situation means that

$$c_1 h_{\bar{z}} < \sqrt{\varepsilon}. \tag{3.35}$$

Let us again try to get (4.5). But this time we need to keep in consideration more terms because ε is not $O(h_{\bar{z}}^2)$ now. Therefore instead of (4.7) we have

$$\tilde{f}(\zeta_0) = -d(\zeta_0)\frac{\partial \tilde{u}}{\partial \xi}(\zeta_0) - \varepsilon\frac{\partial^2 \tilde{u}}{\partial \xi^2}(\zeta_0) - \varepsilon\frac{\partial^2 \tilde{u}}{\partial \eta^2}(\zeta_0). \qquad (3.36)$$

It gives (3.10), (3.11), (3.12), (3.14) to be the same, and (3.13) comes to the following:

$$\frac{1}{2}(\alpha_1\xi_1^2 + \alpha_2\xi_2^2) = -d(\zeta_0)\beta_1 - \varepsilon. \qquad (3.37)$$

Let us repeat considerations (3.15) – (3.27). We obtain the same α_i from (3.20) and β_2 from (3.27). But we get another β_1 and A_{22} :

$$\beta_1 = \frac{1}{8s_{21}}(\eta_1\xi_2^2 - \eta_2\xi_1^2) - \varepsilon/d(\zeta_0), \qquad (3.38)$$

$$A_{22} = -\eta_1\eta_2(\eta_1 - \eta_2)d(\tilde{\zeta})/(8s_{21}) - \varepsilon. \qquad (3.39)$$

From arrangement of ζ_i it follows that

$$-\varepsilon/d(\zeta_0) \le \beta_1 \le \frac{1}{2}\max\{\xi_1, \xi_2\} - \varepsilon/d(\zeta_0). \qquad (3.40)$$

Due to (3.35)

$$|\beta_1| \le h_{\bar{z}}/d(\zeta_0). \qquad (3.41)$$

So, β_1, β_2 are of order $O(h_{\bar{z}})$ and are found by unique way from (3.27), (3.38). After that, one can find α_i from (3.20) with the help of equality $d(\tilde{\zeta}) = d(\beta_1, \beta_2)$. As a result, we obtain

$$\alpha_0\tilde{u}(\zeta_0) + \alpha_1\tilde{u}(\zeta_1) + \alpha_2\tilde{u}(\zeta_2) = \tilde{f}(\zeta_0) + \beta_1\frac{\partial \tilde{f}}{\partial \xi}(\zeta_0) + \beta_2\frac{\partial \tilde{f}}{\partial \eta}(\zeta_0) + A_{22}\frac{\partial^2 \tilde{u}}{\partial \eta^2}(\zeta_0) + O(h_{\bar{z}}^2). \qquad (3.42)$$

In principle, we can make first item in the right-hand side of (3.39) to be small enough due to algorithm of reorientation. For example, let us demand that

$$-\eta_1\eta_2(\eta_1 - \eta_2)d(\tilde{\zeta})/(8s_{21}) \le \varepsilon. \qquad (3.43)$$

It implies

$$-\varepsilon \le A_{22} \le 0. \qquad (3.44)$$

In order to cancel item $A_{22}\partial^2\tilde{u}/\partial\eta^2$ in the right-hand side of (3.42), let us introduce one more triangle with vertices $\zeta_0, \zeta_3, \zeta_4$ (see for Fig. 3); node ζ_3

lies in third quadrant: $\zeta_3 < 0$, $\eta_3 \le 0$; node ζ_4 lies in second one: $\zeta_4 < 0$, $\eta_4 \ge 0$. Consideration like (3.36) – (3.42) gives one more equality

$$\begin{aligned} \alpha'_0\tilde{u}(\zeta_0) + \alpha'_4\tilde{u}(\zeta_4) + \alpha'_3\tilde{u}(\zeta_3) = \tilde{f}(\zeta_0) + \beta'_1\frac{\partial\tilde{f}}{\partial\xi}(\zeta_0) \\ + \beta'_2\frac{\partial\tilde{f}}{\partial\eta}(\zeta_0) + A'_{22}\frac{\partial^2\tilde{u}}{\partial\eta^2}(\zeta_0) + O(h_{\tilde{z}}^2) \end{aligned} \tag{3.45}$$

with coefficients

$$\alpha'_0 = -(\eta_4 - \eta_3)d(\tilde{\zeta}')/(4s_{43}), \tag{3.46}$$
$$\alpha'_4 = -\eta_3 d(\tilde{\zeta}')/(4s_{43}), \tag{3.47}$$
$$\alpha'_3 = \eta_4 d(\tilde{\zeta}')/(4s_{43}), \tag{3.48}$$
$$\beta'_1 = -(\eta_4\xi_3^2 - \eta_3\xi_4^2) - \varepsilon d(\zeta_0)/(8s_{43}), \tag{3.49}$$
$$\beta'_2 = -\eta_4\eta_3(\xi_3 - \xi_4)/(4s_{43}), \tag{3.50}$$
$$A'_{22} = \eta_4\eta_3(\eta_4 - \eta_3)d(\tilde{\zeta}')/(8s_{43}) - \varepsilon, \tag{3.51}$$

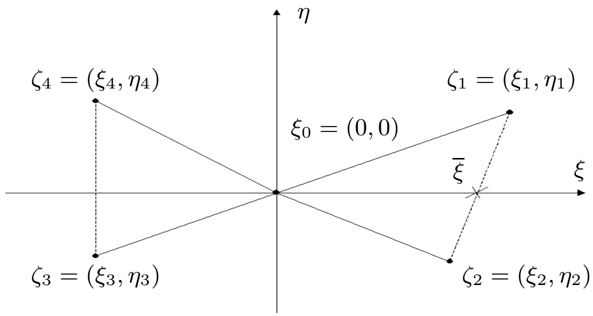

Fig. 3: The local coordinates (ξ, η) and the arrangement of nodes $\zeta_0, ..., \zeta_4$.

where $s_{43} = (\eta_3\xi_4 - \eta_4\xi_3)/2$ is the area of triangle $\Delta\eta_0\eta_4\eta_3$. This time, coefficient A'_{22} consists of two negative items and

$$A'_{22} \le -\varepsilon \tag{3.52}$$

due to arrangement of nodes ζ_4, ζ_3. Let us combine (3.42) and (3.45) with weights δ_1, δ_2 in order to cancel term $\partial^2\tilde{u}/\partial\eta^2(\zeta_0)$:

$$\delta_1 A_{22} + \delta_2 A'_{22} = 0. \tag{3.53}$$

For scaling we take also

$$\delta_1 + \delta_2 = 1. \tag{3.54}$$

This system gives unique solution

$$\delta_1 = A'_{22}/(A'_{22} - A_{22}) > 0,$$
$$\delta_2 = -A_{22}/(A'_{22} - A_{22}) \le 0, \qquad (3.55)$$

when

$$A'_{22} \ne A_{22}. \qquad (3.56)$$

The last is guaranteed when, for example,

$$\eta_3\eta_4 \ne 0 \quad \text{or} \quad \eta_1\eta_2 \ne 0. \qquad (3.57)$$

Due to (3.52), (3.54) we get

$$\sum_{i=0}^{4} \alpha''_i \tilde{u}(\zeta_i) = \tilde{f}(\zeta_0) + \beta''_1 \frac{\partial \tilde{f}}{\partial \xi}(\zeta_0) + \beta''_2 \frac{\partial \tilde{f}}{\partial \eta}(\zeta_0) + O(h^2_{\bar{z}}) \qquad (3.58)$$

where

$$\alpha''_0 = \delta_1\alpha_0 + \delta_2\alpha'_0 > 0,\ \alpha''_1 = \delta_1\alpha_1 \le 0,$$
$$\alpha''_2 = \delta_1\alpha_2 \le 0,\ \alpha''_3 = \delta_2\alpha'_3 \le 0,\ \alpha''_4 = \delta_2\alpha'_4 \le 0, \qquad (3.59)$$
$$\beta''_1 = \delta_1\beta_1 + \delta_2\beta'_1,\ \beta''_2 = \delta_1\beta_2 + \delta_2\beta'_2.$$

The signs of α''_i provide the inverse monotonicity of difference operator in the left-hand side of (3.58).

4 The algorithm for the orientation strengthening of the difference grid

Let us consider an arbitrary opened limited, and connected polygon $\Omega \in R^2$. We construct its triangulation $\mathcal{J}$, i.e., we cut this polygon into the finite number of opened triangles T_i, $i = 1, ..., m$, so that their closure $\overline{T}_i$ cover $\overline{\Omega}$:

$$\overline{\Omega} = \bigcup_{i=1}^{m} \overline{T}_i. \qquad (4.1)$$

This triangulation should be consistent, i.e., any two different closed triangles $\overline{T_i}$ and $\overline{T_j}$ from $\mathcal{J}$, $i \ne j$, either have no common points, or only one common vertex, or have the whole common side.

Let us denote by $\overline{\Omega}_h$ a set of all vertices of triangulation triangles, which are called by nodes. Suppose that

$$\Omega_h = \overline{\Omega}_h \cap \Omega, \quad \Gamma_h = \overline{\Omega}_h \cap \Gamma. \qquad (4.2)$$

Our goal is to describe the algorithm of triangles reconstruction in order to decrease the computational diffusion across characteristic lines of difference analogue to necessary limits. In section 3, we introduce a special local value to control it.

There are many ways of grid construction of different complexity. The grid are condensing in the required subdomains or oriented with some method. But all of them are connected either with the new nodes addition, or with the inner nodes coordinates modification.

We propose the algorithm that does not change the coordinates of inner nodes, but it makes better the desired quality of triangulation due to reconnection of the nodes among themselves.

Now, we consider the initial consistent triangulation $\mathcal{J}'$. One of the algorithm cycles consists of step-by-step sorting out of inner apexes $z_i \in \mathcal{J}' \cap \Omega$, $i = 1, ..., n$, by means of possible triangles reconstruction. Let us describe one step of this algorithm.

Let i be inner node $z_i = (x_i, y_i)$ of the consistent triangulation $\mathcal{J}'$ with anticharacteristic vector $-t(z_i)$, which we reconstruct to perform the inequality

$$\max_{z_i \in \mathcal{J}' \cap \Omega} Kr(z_i) \le \delta \tag{4.3}$$

with some constant δ.

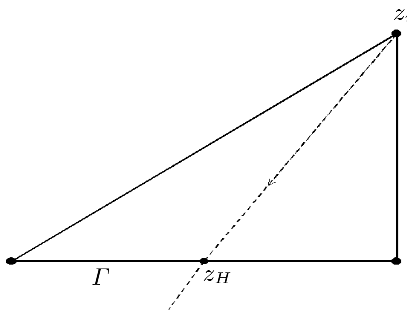

Fig. 4: The anticharacteristic direction crossing the boundary.

1. If this vector is directed along one of the triangle sides, with the origin in this vertex, then local criterion $(K_r(z_i) = 0)$ is considered to be valid and we complete the step without changing the triangulation, i.e., the result $\mathcal{J}''$ of this step coincides with $\mathcal{J}'$.

2. If this coincidence (which is unlikely in real problems) does not take place, then there is triangle $T_k \in \mathcal{J}'$ with vertex z_i, for which vector $-t(z_i)$ is enclosed between its sides. Let construct a ray in this direction to cross a side of this triangle, which is opposite to vertex z_i.

Futher, there are two variants.

2.1) The triangle side crossed lies on the boundary Γ (Fig. 4). In this case we add a new node z_H to Γ_h , which is the intersection point of the constructed ray and boundary Γ. In this case we obtain the ideal situation, $Kr(z_i) = 0$.

2.2) The triangle side cross is the inner one (Fig. 5). Since triangulation is consistent, there exists one more triangle with the same side.

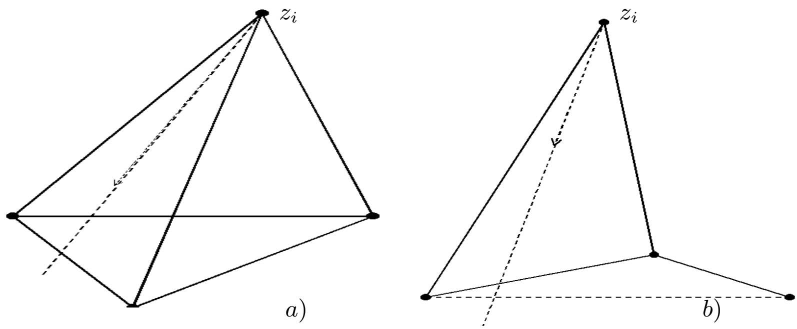

Fig. 5: The anticharacteristic direction crossing the inner side.

Further, there are two variant as well.

2.2.1) The obtained quadrangle is convex (Fig. 5.a). From two available variants we choose such that gives criterion $Kr(z_i)$ to be smaller.

2.2.2) The obtained quadrangle is not convex (Fig. 5.b). Then we complete the step without changing the triangulation.

In this way, the process is periodically repeated for all nodes z of Ω_h where $Kr(z) > \delta$. It should be pointed out that the effectiveness of algorithm will be better, if we move forward by front through inner nodes along the convective flow.

5 The numerical experiment

For the numerical experiment we considered problem (2.1) – (2.2) with coefficients $b_1 = -1, \quad b_2 = 0.7$. The function g is equal to zero on the boundary Γ except for two sections

$$\Gamma_1 = \{(x, y) : x = 0,\ y \in [35/40, 39/40]\}$$

and

$$\Gamma_2 = \{(x, y) : x = 1,\ y \in [1/40, 5/40]\}$$

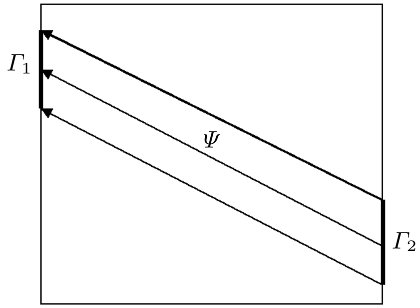

Fig. 6: The characteristics of the reduced equation.

(see Fig. 6), where g equals 1. The right-hand side is identically equal to zero on the Ω. For $\varepsilon = 0$ the exact solution of reduced problem is the function u_0 which is equal to 1 in band $\overline{\Psi}$ and 0 outside it (see Fig. 7, 8). The band $\overline{\Psi}$ represents the parallelogram with sides Γ_1 and Γ_2.

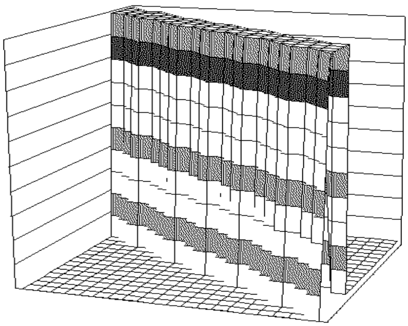

Fig. 7: The exact solution.

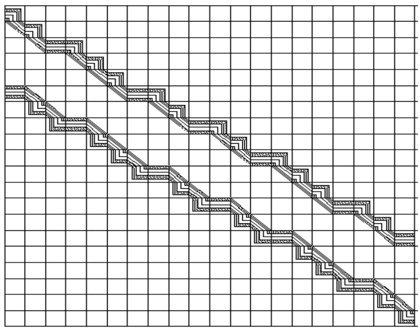

Fig. 8: The isolines of exact solution.

In square $\Omega = \{\,(x, y) : 0 < x < 1,\ 0 < y < 1\,\}$ we build the uniform triangulation with the mesh-size $h = 1/n$ by means of two families of lines $x_i = ih$, $y_j = jh$, $i, j = 1, ..., n-1$, and then construct the diagonals in the obtained elementary squares with angle $\pi/4$ to axis Ox.

Then we build the grid aproximation in the following way. To aproximate items Δu we use on the uniform five-point stencil "cross". As the aproximation of item $b_1\,\partial u/\partial x + b_2\,\partial u/\partial y$, we realize it on the constructed triangulation.

This triangulation is unsuccessful (Fig. 9) in term of the value of orientation. Solving the problem for $n = 40$ with this triangulation, we do not

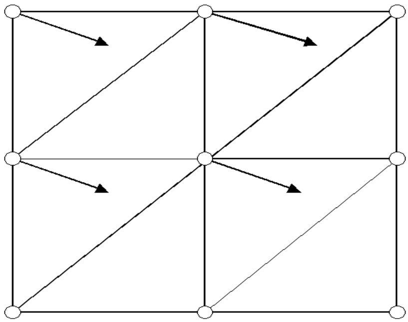

Fig. 9: The inital triangulation.

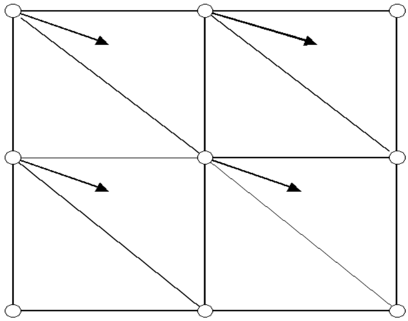

Fig. 10: The grid after the first reconstruction.

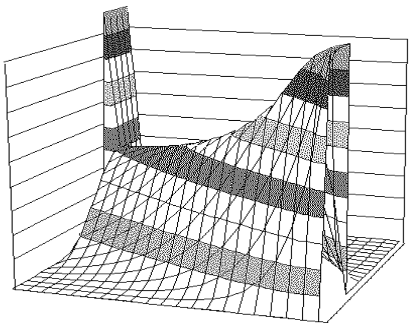

Fig. 11: The numerical solution on the inital grid.

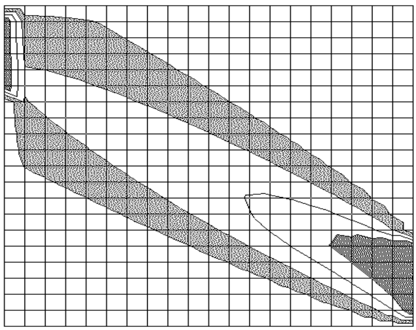

Fig. 12: The isolines of numerical solution on the inital grid.

obtain even the qualitative similarity solution. The considerable "transversal" calculation diffusion appeares which washes out the solution (Fig. 12), and obtained error equals 60% (Fig. 11).

Further the first reconstruction of grid is made, which we implement according to section 5. It only reorients some diagonals without coordinates modification of inner nodes (Fig. 10). Solving again the problem for $n = 40$ with this triangulation, we obtain the considerable improvement of the solution quality. The essential decrease of computing diffusion took place (Fig. 14), and the obtained error equals 20% (Fig. 13).

After the second application of the reorientation algorithm the new nodes on the boundary of domain appear, which do not involve the increase of the unknown values in consequence of known boundary conditions. Apart from that, the recombination of inner grid nodes with each other (Fig. 15) consequently decreases $Kr(z_i)$ in every inner nodes. The obtained

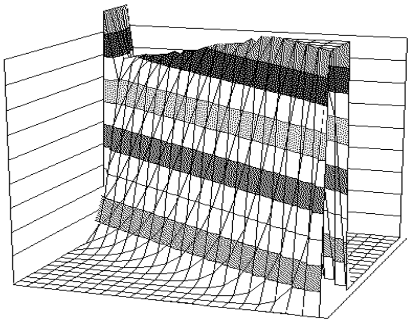

Fig. 13: The numerical solution after the first grid reconstruction.

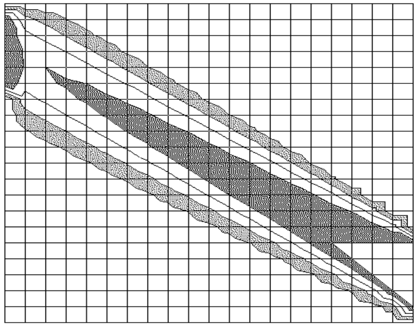

Fig. 14: The isolines of numerical solution after the first grid reconstruction.

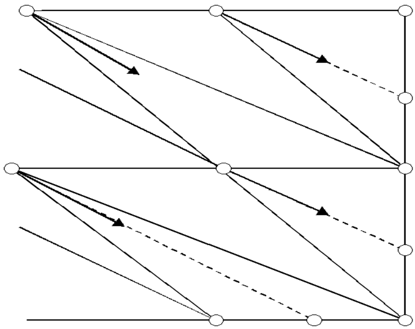

Fig. 15: The grid after the second reconstruction.

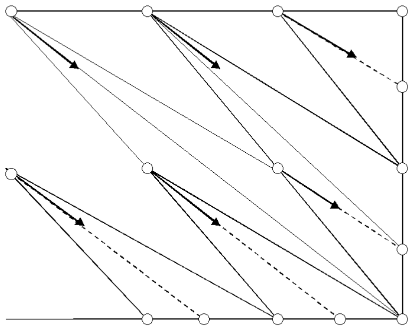

Fig. 16: The grid after the third reconstruction.

error is not greater than 10% (Fig. 17) and we note some decrease of the solution wash-out (Fig. 18).

After the third reconstruction of grid (Fig. 16), we obtain the considerable improvement of solution. Apart from similar qualitative behavior of the solution (Fig. 19), we also obtain good quantitative similarity.

Thus, this numerical experiment illustrates the successive improvement of numerical solution on first three stages of the grid reconstruction due to strengthening of the orientation along the characteristic curves.

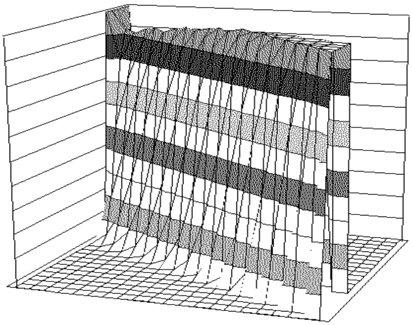

Fig. 17: The numerical solution after the second grid reconstruction.

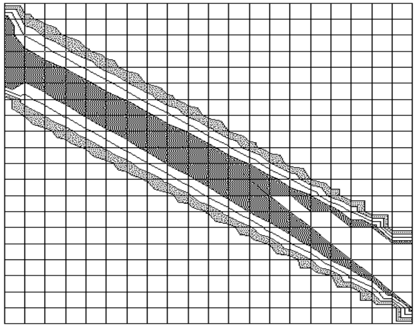

Fig. 18: The isolines of numerical solution after the second grid reconstruction.

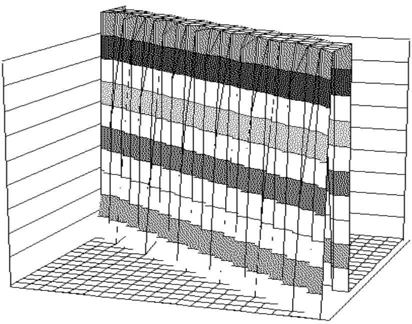

Fig. 19: The numerical solution after the third grid reconstruction.

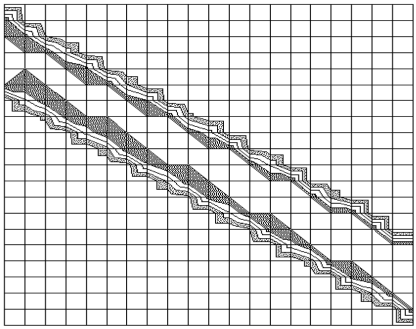

Fig. 20: The isolines of numerical solution after the third grid reconstruction.

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