A nonuniform difference scheme with fourth order of accuracy in a domain with smooth boundary

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Introduction

The present paper continues a series of works devoted to construction and justification *nonuniform* difference schemes of higher degrees of accuracy. Two-dimensional boundary-value problem for elliptic type equation in a domain with smooth curvilinear boundary is considered. The main idea of construction of such scheme is similar to that in the papers [1], [11], where it is stated for the same equation in a rectangle. The transition to curvilinear boundary required either to solve the question on special approximation of boundary values or to re-construct the grid equations on non-standard stencils near the boundary. Both approaches were used as applied to Richardson extrapolation in [2], [3] and [4], [5], [6], respectively. The first, although leads to required result, gives extensive stencils; the second, being more complicated in theoretical respect, gives more compact stencils of difference equations near the boundary, so it appeared to be more preferable.

As it is in one-dimensional case, the difference scheme inside the domain is similar in structure to the equations of the method of extrapolated equations by U. Rüde [7] for finite elements. But near the boundary the equations appear to be different. The justification of accuracy here is also different from [7] and based on the maximum principle for a system of linear algebraic equations equivalent to the difference scheme.

Let recall that the term nonuniform scheme was introduced in [8] and used in [1] due to two different rules of construction of grid equations in neighbouring nodes as distinct from uniform schemes [9], when the rule of construction is the same for all nodes of the grid, at least inside the domain.

1 Boundary-value problem

Let Ω be a limited domain in R^2 with smooth boundary Γ (i.e. of the class C^1). Consider a boundary-value problem

$$-\Delta u + du = f \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = g \quad \text{on} \quad \Gamma \tag{1.2}$$

with continuous on $\overline{\Omega}$ functions d, f and continuous on Γ function g, and

$$d \geq 0 \quad \text{on} \quad \overline{\Omega}. \tag{1.3}$$

These conditions ensure unique solvability of the problem. Suppose that the solution is smooth enough:

$$u \in C^6(\overline{\Omega}). \tag{1.4}$$

2 Construction of the difference grid and classification of its nodes

Likewise in [6], suppose that the domain Ω is located within the square $\{(x, y) : \ 0 \leq x \leq 1, \ \ 0 \leq y \leq 1\}$. Cover it with a square grid with the step $h = 1/N$, formed by the lines $x_i = ih$ and $y_j = jh$, where $i, j = 0, 1, \ldots, N$ and N is integer. Let call *nodes* the points of intersection of these lines. A node z_{ij} is called *inner*, if $z_{ij} \in \Omega$. Denote the set of all inner nodes by ω_h.

Each line of the grid x_i or y_j which intersects Ω also intersects the boundary Γ. Due to smoothness of the boundary the intersection with the domain consists of certain number of intervals. Let call the end of these intervals *boundary nodes* in direction x (or y), if the line being considered is parallel to the coordinate axis Ox (respectively Oy). The set of all boundary nodes in direction x denote by $\gamma_{h,x}$, and the set of all boundary nodes in direction y denote by $\gamma_{h,y}$. Also denote

$$\gamma_h = \gamma_{h,x} \cup \gamma_{h,y} \quad \text{and} \quad \overline{\omega}_h = \omega_h \cup \gamma_h.$$

For convenience, let divide the set ω_h into four subsets

$$\omega_{00} = \{z_{ij} : \ z_{ij} \in \omega_h, \ i \text{ is even}, \ j \text{ is even}\};$$
$$\omega_{01} = \{z_{ij} : \ z_{ij} \in \omega_h, \ i \text{ is even}, \ j \text{ is odd}\};$$
$$\omega_{10} = \{z_{ij} : \ z_{ij} \in \omega_h, \ i \text{ is odd}, \ j \text{ is even}\};$$
$$\omega_{11} = \{z_{ij} : \ z_{ij} \in \omega_h, \ i \text{ is odd}, \ j \text{ is odd}\}.$$

For each inner node $z_{ij} = (x_i, y_j)$ introduce two definitions of the distance to the boundary Γ which is parallel to two coordinate axes:

$$\rho_1(x_i, y_j) = \min_{(x,y_j)\in\Gamma} |x_i - x|,$$
$$\rho_2(x_i, y_j) = \min_{(x_i,y)\in\Gamma} |y_j - y|.$$

With the help of these definitions introduce a classification of the inner nodes ω_h. Geometric illustration of this classification is given at the end of the paper for a concrete numerical example. Denote by $\gamma^1_{1,h}$ *the set of inner irregular nodes of the first type*, for which only one of the distances ρ_1 or ρ_2 is less than h, and the another is greater than or equal to $2h$:

$$\gamma^1_{1,h} = \{z_{ij} : z_{ij} \in \ \omega_h,\ (\rho_1(z_{ij}) < h)\ \&\ (\rho_2(z_{ij}) \ge 2h)$$
$$\text{or } (\rho_1(z_{ij}) \ge 2h)\ \&\ (\rho_2(z_{ij}) < h)\}.$$

By $\gamma^2_{1,h}$ let denote *the set of inner irregular nodes of the second type*, for which both the distances ρ_1 and ρ_2 are less than h:

$$\gamma^2_{1,h} = \{z_{ij} :\ z_{ij} \in \omega_h,\ (\rho_1(z_{ij}) < h)\ \&\ (\rho_2(z_{ij}) < h)\}.$$

Respectively, by $\gamma^3_{1,h}$ let denote the set of *inner irregular nodes of the third type*, for which at least one adjacent node $z_{i,j\pm1}$, $z_{i\pm1,j}$ belongs to $\gamma^2_{1,h}$:

$$\gamma^3_{1,h} = \{z_{ij} : z_{ij} \in \ \ \omega_h,\ z_{i+1,j} \in \gamma^2_{1,h} \text{ or } z_{i-1,j} \in \gamma^2_{1,h}$$
$$\text{or } z_{i,j+1} \in \gamma^2_{1,h} \text{ or } z_{i,j-1} \in \gamma^2_{1,h}\}.\ ^{*)}$$

By $\gamma^{out}_{1,h}$ denote the set of external irregular nodes z_{ij}, for which at least one adjacent node $z_{i\pm1,j}$, $z_{i,j\pm1}$ belongs to $\gamma^1_{1,h} \cup \gamma^3_{1,h}$:

$$\gamma^{out}_{1,h} = \{z_{ij} : z_{ij} \notin \ \ \overline{\Omega},\ z_{i+1j} \in \gamma^1_{1,h} \cup \gamma^3_{1,h} \text{ or } z_{i-1j} \in \gamma^1_{1,h} \cup \gamma^3_{1,h}$$
$$\text{or } z_{ij+1} \in \gamma^1_{1,h} \cup \gamma^3_{1,h} \text{ or } z_{ij-1} \in \gamma^1_{1,h} \cup \gamma^3_{1,h}\}.$$

Now, let classify the nodes from ω_{00} near the boundary, which have not come into $\gamma^1_{1,h}$, $\gamma^2_{1,h}$ or $\gamma^3_{1,h}$. Denote by $\gamma_{2,h}$ the set of multiple nodes near the boundary, for which at least one of the distances ρ_1 or ρ_2 is less than $3h$, i.e.,

$$\gamma_{2,h} = \ \{z_{ij} : z_{ij} \in \omega_{00} \setminus (\gamma^1_{1,h} \cup \gamma^2_{1,h} \cup \gamma^3_{1,h}),$$
$$(\rho_1(z_{ij}) < 3h) \text{ or } (\rho_2(z_{ij}) < 3h)\}.$$

By $\gamma_{2,h}^1$ denote a subset of nodes from $\gamma_{2,h}$, for which at least one of the distances ρ_1 or ρ_2 is less than $2h$:

$$\gamma_{2,h}^1 = \{z_{ij} : z_{ij} \in \gamma_{2,h},\ (\rho_1(z_{ij}) < 2h) \text{ or } (\rho_2(z_{ij}) < 2h)\}.$$

By $\gamma_{2,h}^2$ denote a subset of nodes from $\gamma_{2,h}$, for which both the distances ρ_1 and ρ_2 are greater than $2h$:

$$\gamma_{2,h}^2 = \{z_{ij} : z_{ij} \in \gamma_{2,h},\ (\rho_1(z_{ij}) > 2h)\ \&\ (\rho_2(z_{ij}) > 2h)\}.$$

And, finally, by $\gamma_{2,h}^3$ let denote a subset of nodes from $\gamma_{2,h}$, for which only one of the distances ρ_1 or ρ_2 is greater than $2h$, and the another is greater than $3h$:

$$\gamma_{2,h}^3 = \gamma_{2,h} \setminus (\gamma_{2,h}^1 \cup \gamma_{2,h}^2).$$

For convenience of subsequent consideration, let divide $\gamma_{1,h}^3$ into three subsets:

1) $\gamma_{1,h}^{31}$ consists of nodes whose both adjacent nodes belong to $\gamma_{1,h}^2$;

2) $\gamma_{1,h}^{32}$ consists of nodes whose one adjacent node belongs to $\gamma_{1,h}^2$, and the other one belongs to $\gamma_{1,h}^{out}$;

3) $\gamma_{1,h}^{33} = \gamma_{1,h}^3 \setminus (\gamma_{1,h}^{31} \cup \gamma_{1,h}^{32})$.

Make a classification of regular nodes. Let call a node *regular of the first kind,* if it belongs to $\omega_h \setminus \omega_{00}$ and is not included in $\gamma_{1,h}^1$, $\gamma_{1,h}^2$, $\gamma_{1,h}^3$; denote the set of such nodes by

$$\omega_{h,1}^r = \omega_h \setminus (\omega_{00} \cup \gamma_{1,h}^1 \cup \gamma_{1,h}^2 \cup \gamma_{1,h}^3).$$

Let call a node *regular of the second kind,* if it belongs to ω_{00}, but is not included in $\gamma_{2,h}$; denote the set of such nodes by

$$\omega_{h,2}^r = \omega_{00} \setminus (\gamma_{2,h} \cup \gamma_{1,h}^1 \cup \gamma_{1,h}^2 \cup \gamma_{1,h}^3).$$

The totality of regular nodes denote by $\omega_h^r = \omega_{h,1}^r \cup \omega_{h,2}^r$, and the nodes $\omega_h^{ir} = \omega_h \setminus \left\{\omega_h^r \cup \gamma_{1,h}^2\right\}$ let call irregular one.

For an arbitrary function v defined on a set D (finite or infinite) let introduce the denotation

$$\|v\|_{\infty,D} = \sup_D |v|.$$

3 Interpolation formula

For interpolation of boundary values we will use the interpolation formulas of two forms, selected in each case for ensuring stability (the stencils are shown in Fig. 1).

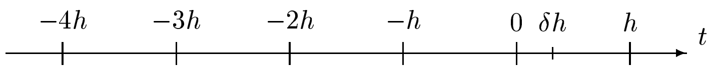


Fig. 1: Stencil of Lagrange interpolation formula; $0 < \delta \leq 1$.

Let the function $v(t) \in C^4[-3h, \delta h]$. The first of the formulas approximates the value $v(0)$ through the values of the same function in the points $-h$, $-2h$, $-3h$ and δh ($0 < \delta \leq 1$):

$$\begin{aligned} v(0) &\approx \varphi_1(v(-h), v(-2h), v(-3h), v(\delta h)) \\ &= \frac{3\delta}{\delta+1} v(-h) - \frac{3\delta}{\delta+2} v(-2h) + \frac{\delta}{\delta+3} v(-3h) + \varphi_{1f}(\delta) v(\delta h) \end{aligned} \tag{3.1}$$

where $\varphi_{1f}(\delta) = 6/((\delta+1)(\delta+2)(\delta+3))$.

Now, let supplement the definition of the function v to the right from δh with a segment of Taylor series with respect to δh up to the fourth derivative inclusive. Let keep the denotation $v(t)$ for the supplement, and note that it belongs to $C^4[-2h, h]$, and

$$\|v^{(4)}\|_{\infty,[-2h,h]} = \|v^{(4)}\|_{\infty,[-2h,\delta h]}.$$

The second formula expresses $v(h)$ through four values in the points 0, $-h$, $-2h$, and δh:

$$\begin{aligned} v(h) &\approx \varphi_2(v(0), v(-h), v(-2h), v(\delta h)) \\ &= -\frac{3(1-\delta)}{\delta} v(0) + \frac{3(1-\delta)}{\delta+1} v(-h) - \frac{1-\delta}{\delta+2} v(-2h) + \varphi_{2f}(\delta) v(\delta h) \end{aligned} \tag{3.2}$$

where $\varphi_{2f}(\delta) = 6/(\delta(\delta+1)(\delta+2))$.

Let recall [10] that interpolation over four nodes gives result with fourth order of accuracy in the following form:

$$\max\{|v(0) - \varphi_1|, |v(h) - \varphi_2|\} \leq c h^4 \|v^{(4)}\|_{\infty,[-3h,\delta h]} \tag{3.3}$$

with the constant c independent of h, $v(t)$, and $\delta \in (0, 1]$.

4 Construction of difference approximation

For difference approximation of the equation (1.1) introduce the following operators:

$$\begin{aligned} L^h v(x,y) \equiv& (v(x-h,y) + v(x+h,y) + v(x,y-h) + v(x,y+h) - \\ & 4v(x,y))/h^2 + d(x,y)v(x,y), \end{aligned} \tag{4.1}$$

$$\begin{aligned} L^{2h} v(x,y) \equiv& (v(x-2h,y) + v(x+2h,y) + v(x,y-2h) + v(x,y+2h) - \\ & 4v(x,y))/(4h^2) + d(x,y)v(x,y). \end{aligned} \tag{4.2}$$

Start the construction of grid equations with the regular nodes. Let $z_{ij} \in \omega^r_{h,1}$. Then the difference approximation is performed as a standard five-point equation

$$L^h u^h(z_{ij}) = f(z_{ij}), \quad z_{ij} \in \omega^r_{h,1}, \tag{4.3}$$

on the stencil small cross (see Fig. 2.a)). But if z_{ij} is a regular node of the second kind, i.e., $z_{ij} \in \omega^r_{h,2}$, then the difference approximation is taken as nine-point equation on the stencil large cross (see. Fig. 2.b)):

$$L^h u^h(z_{ij}) - L^{2h} u^h(z_{ij}) = 0, \quad z_{ij} \in \omega^r_{h,2}. \tag{4.4}$$

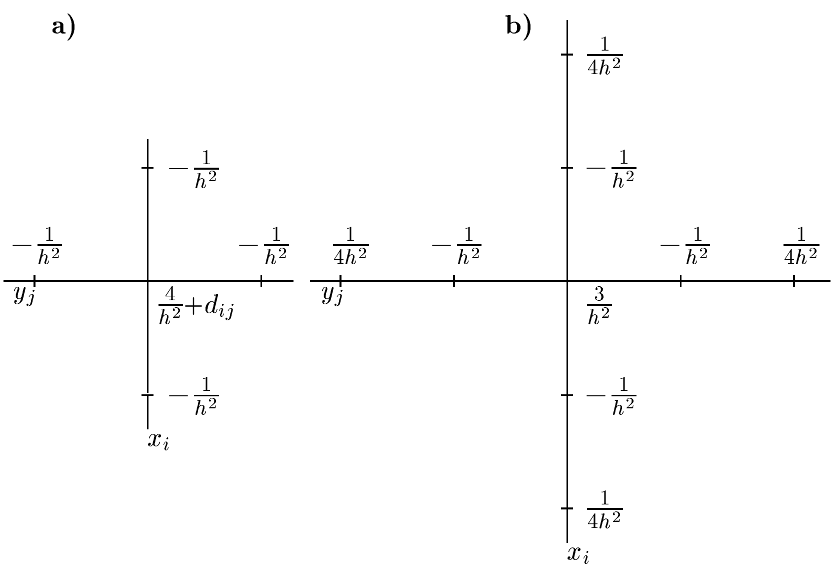


Fig. 2: Stencils **a)** small cross and **b)** large cross with the values of coefficients of the operators L^h and $L^h - L^{2h}$, respectively.

Consider the construction of grid equation in irregular nodes. Let $z_{ij} \in \gamma^1_{1,h}$ and one node of the stencil small cross, for instance, $z_{i,j+1}$ does not belong to Ω (see Fig. 3.a)). Denote by s_{ij} the point of intersection of the boundary Γ with the segment $[z_{ij}, z_{i,j+1}]$. At the beginning assume that the solution $u(x, y)$ is determined in the point $z_{i,j+1}$ and write down an ordinary five-point equation (4.3). Then construct interpolation formula (3.2) for the function $u(x,y)$ with respect to y coordinate, directing the axis $0t$ from z_{ij} into $z_{i,j+1}$ and assuming δ be equal to the distance δ_y from z_{ij} to s_{ij}, i.e., $\delta_y = \rho_2(z_{ij})$. As a result, we obtain five-point grid equation with *the first asymmetric T-shaped stencil* :

$$\begin{aligned}&\left(\frac{4}{h^2} + d(z_{ij}) + \frac{3(1-\delta_y)}{\delta_y h^2}\right) u^h(z_{ij}) - \left(\frac{1}{h^2} + \frac{3(1-\delta_y)}{(1+\delta_y)h^2}\right) u^h(z_{i,j-1})\\ &+\frac{1-\delta_y}{(2+\delta_y)h^2} u^h(z_{i,j-2}) - \frac{1}{h^2} u^h(z_{i+1,j}) - \frac{1}{h^2} u^h(z_{i-1,j}) \qquad (4.5)\\ &= f(z_{ij}) + \frac{1}{h^2}\varphi_{2f}(\delta_y) g(s_{ij})\end{aligned}$$

where the boundary condition (2) is used in the form $u(s_{ij}) = g(s_{ij})$.

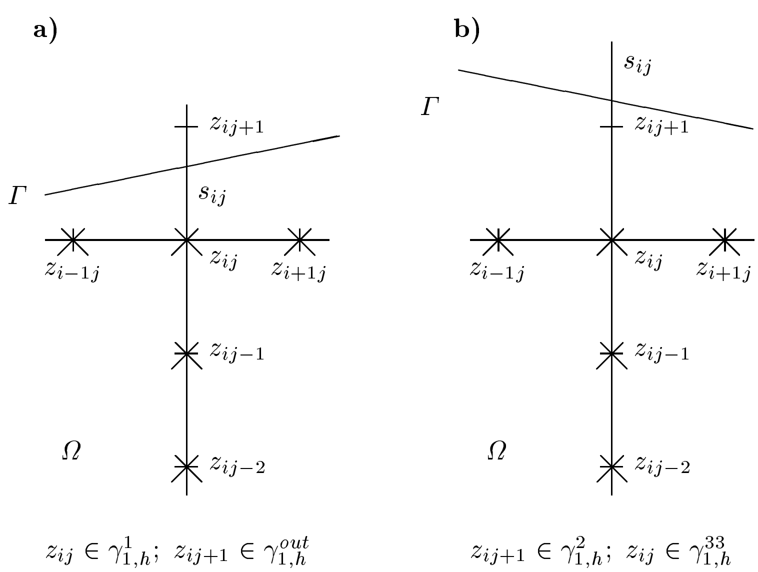


Fig. 3: The first (a) and the second (b) T-shaped asymmetric stencils. Cross sign marks the nodes of corresponding stencils.

Remark. Construction of grid equations is not performed in the nodes $z_{ij} \in \gamma^2_{1,h}$. An attempt of double application of the above method in these nodes (elimination of external nodes by means of mean value formula (3.2)) gives grid equations which do not provide sufficient conditions for comparison theorems and the proof of stability. Therefore the authors refused from use of grid equations in nodes of the set $\gamma^2_{1,h}$. Accordingly, the resulting system of equations should not contain variables $u^h(z_{ij})$ with arguments from $\gamma^2_{1,h}$.

Taking into account the above remark, it is necessary to exclude the values in the nodes $\gamma^2_{1,h}$ from the other grid equations. Three variants are possible when one or two nodes of the stencil small cross belong to $\gamma^2_{1,h}$, and there are two variants when one or two nodes of the stencil large cross belong to $\gamma^2_{1,h}$.

Consider these variants.

1) Suppose that $z_{ij} \in \gamma^{33}_{1,h}$ and one node of the stencil small cross, for instance, $z_{i,j+1}$ belongs to $\gamma^2_{1,h}$ (see Fig. 3.b)). Denote by s_{ij} the point of intersection of the boundary Γ with the ray $[z_{ij}, z_{i,j+1})$. At the beginning assume that the solution $u(x,y)$ is determined in the point $z_{i,j+1}$ and write down an ordinary five-point equation (4.3). Then construct interpolation formula (3.1) for the function $u(x,y)$ with respect to y coordinate, directing the axis $0t$ from $z_{i,j+1}$ into s_{ij} and assuming δ be equal to the distance δ_y from $z_{i,j+1}$ to s_{ij}, i.e., $\delta_y = \rho_2(z_{i,j+1})$. As a result, we obtain five-point grid equation with *the second asymmetric T-shaped stencil*:

$$\begin{aligned}
&\left(\frac{4}{h^2} + d(z_{ij}) - \frac{3\delta_y}{(\delta_y+1)h^2}\right) u^h(z_{ij}) - \left(\frac{1}{h^2} - \frac{3\delta_y}{(2+\delta_y)h^2}\right) u^h(z_{i,j-1}) \\
&-\frac{\delta_y}{(3+\delta_y)h^2} u^h(z_{i,j-2}) - \frac{1}{h^2} u^h(z_{i+1,j}) - \frac{1}{h^2} u^h(z_{i-1,j}) \\
&= f(z_{ij}) + \frac{1}{h^2}\varphi_{1f}(\delta_y) g(s_{ij}).
\end{aligned} \tag{4.6}$$

2) Let $z_{ij} \in \gamma^{31}_{1,h}$ and two nodes of the stencil small cross, for instance, $z_{i,j+1}$ and $z_{i+1,j}$ belong to $\gamma^2_{1,h}$ (see Fig. 4.). Both the values $u^h(z_{i,j+1})$ and $u^h(z_{i+1,j})$ are eliminated by means of formula (3.1). As a result, we obtain five-point grid equation with *the first asymmetric Γ-shaped stencil* (see Fig. 4.):

$$\begin{aligned}
&\left(\frac{4}{h^2} + d(z_{ij}) - \frac{3\delta_x}{(\delta_x+1)h^2} - \frac{3\delta_y}{(\delta_y+1)h^2}\right) u^h(z_{ij}) \\
&- \left(\frac{1}{h^2} - \frac{3\delta_x}{(2+\delta_x)h^2}\right) u^h(z_{i-1,j}) - \left(\frac{1}{h^2} - \frac{3\delta_y}{(2+\delta_y)h^2}\right) u^h(z_{i,j-1}) \\
&-\frac{\delta_x}{(3+\delta_x)h^2} u^h(z_{i-2,j}) - \frac{\delta_y}{(3+\delta_y)h^2} u^h(z_{i,j-2}) \\
&= f(z_{ij}) + \frac{1}{h^2}\varphi_{1f}(\delta_x) g(s_{ij_x}) + \frac{1}{h^2}\varphi_{1f}(\delta_y) g(s_{ij_y})
\end{aligned} \tag{4.7}$$

where $\delta_x = \rho_1(z_{i+1,j})$ and $\delta_y = \rho_2(z_{i,j+1})$.

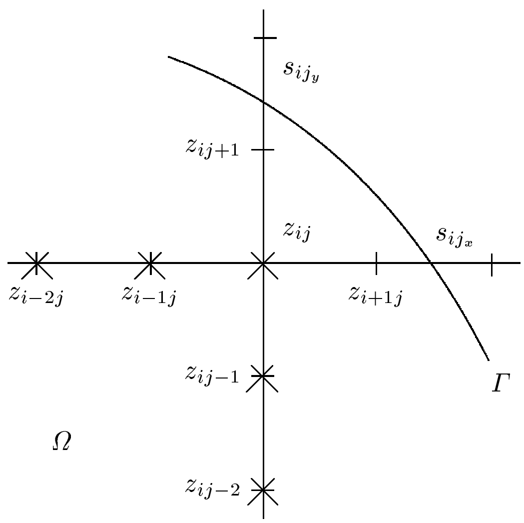


Fig. 4: The first Γ-shaped asymmetric stencil.
Cross sign marks the nodes of the stencil. $z_{i,j+1} \in \gamma^2_{1,h}$; $z_{i+1,j} \in \gamma^2_{1,h}$; $z_{ij} \in \gamma^{31}_{1,h}$.

3) Let $z_{ij} \in \gamma^{32}_{1,h}$, one node of the stencil small cross, for instance, $z_{i,j+1}$ belong to $\gamma^2_{1,h}$ and the second, for instance, $z_{i+1,j}$ belong to $\gamma^{out}_{1,h}$ (see Fig. 5.). Both the values $u^h(z_{i,j+1})$ and $u^h(z_{i+1,j})$ are eliminated by means of corresponding formula (3.1) or (3.2). As a result, we obtain five-point grid equation with *the second asymmetric Γ-shaped stencil* (see Fig. 5.):

$$\begin{aligned}
&\left(\frac{4}{h^2} + d(z_{ij}) + \frac{3(1-\delta_x)}{\delta_x h^2} - \frac{3\delta_y}{(\delta_y+1)h^2}\right) u^h(z_{ij}) \\
&-\frac{\delta_y}{(3+\delta_y)h^2} u^h(z_{i,j-2}) - \left(\frac{1}{h^2} + \frac{3(1-\delta_x)}{(1+\delta_x)h^2}\right) u^h(z_{i-1,j}) \\
&-\left(\frac{1}{h^2} - \frac{3\delta_y}{(2+\delta_y)h^2}\right) u^h(z_{i,j-1}) + \frac{1-\delta_x}{(2+\delta_x)h^2} u^h(z_{i-2,j}) \\
&= f(z_{ij}) + \frac{1}{h^2}\varphi_{2f}(\delta_x) g(s_{ij_x}) + \frac{1}{h^2}\varphi_{1f}(\delta_y) g(s_{ij_y})
\end{aligned} \tag{4.8}$$

where $\delta_x = \rho_1(z_{ij})$ and $\delta_y = \rho_2(z_{i,j+1})$.

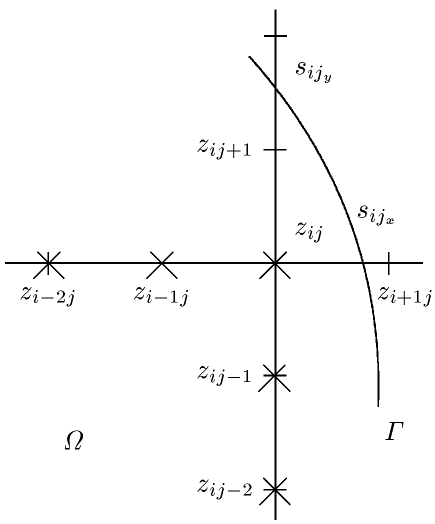


Fig. 5: The second Γ-shaped asymmetric stencil.

Cross sign marks the nodes of the stencil. $z_{i,j+1} \in \gamma^2_{1,h}$; $z_{i+1,j} \in \gamma^{out}_{1,h}$; $z_{ij} \in \gamma^{32}_{1,h}$.

Consider the equations in irregular nodes z_{ij} belonging to $\gamma_{2,h}$. There are two variants when one or two points of the stencil large cross belong to $\gamma^2_{1,h}$. Let $z_{ij} \in \gamma^3_{2,h}$ and $z_{i-2,j} \in \gamma^2_{1,h}$ (see Fig. 6.). Denote by s_{ij} the point of intersection of the boundary Γ and the ray $(z_{i-2,j}, z_{ij}]$. At the beginning assume that the solution $u(x,y)$ is determined in the point $z_{i-2,j}$ and write down five-point equation:

$$\begin{aligned} &\left(\frac{1}{h^2} + d(z_{ij})\right) u^h(z_{ij}) - \frac{1}{4h^2} u^h(z_{i-2,j}) - \frac{1}{4h^2} u^h(z_{i+2,j}) \\ &-\frac{1}{4h^2} u^h(z_{i,j-2}) - \frac{1}{4h^2} u^h(z_{i,j+2}) = f(z_{ij}). \end{aligned} \tag{4.9}$$

Then construct interpolation formula (3.1) for the function $u(x,y)$ with respect to y coordinate, directing the axis $0t$ from z_{ij} into $z_{i-2,j}$ and assuming δ be equal to the distance δ_y from $z_{i-2,j}$ to s_{ij}. As a result, we obtain six-point grid equation with *the first asymmetric X-shaped stencil*:

$$\begin{aligned}
&\left(\frac{1}{h^2}+\frac{3\delta_y}{(\delta_y+2)4h^2}+d(z_{ij})\right)u^h(z_{ij})-\frac{3\delta_y}{(\delta_y+1)4h^2}u^h(z_{i-1,j})\\
&-\frac{\delta_y}{(\delta_y+3)4h^2}u^h(z_{i+1,j})-\frac{1}{4h^2}u^h(z_{i+2,j})-\frac{1}{4h^2}u^h(z_{i,j+2}) \qquad (4.10)\\
&-\frac{1}{4h^2}u^h(z_{i,j-2})=f(z_{ij})+\frac{1}{4h^2}\varphi_{1f}(\delta_y)g(s_{ij}).
\end{aligned}$$

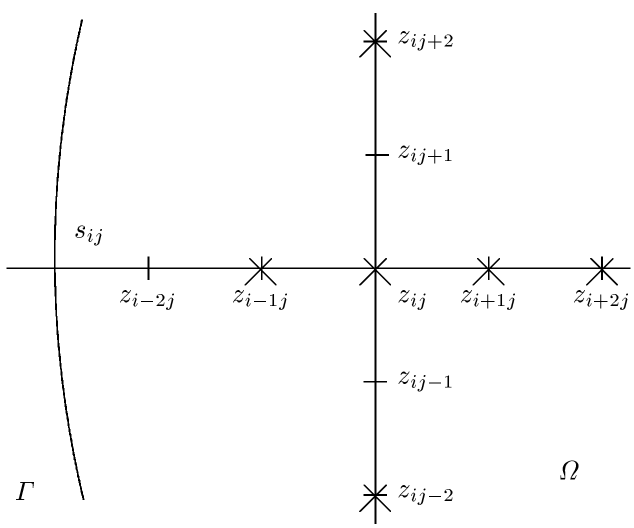


Fig. 6: The first X-shaped asymmetric stencil.

Cross-sign marks the nodes of the stencil. $z_{ij}\in\gamma^3_{2,h}$; $z_{i-2,j}\in\gamma^2_{1,h}$.

Consider the second variant. Let $z_{ij}\in\gamma^2_{2,h}$ and $z_{i-2,j}\in\gamma^2_{1,h}$, $z_{i,j+2}\in\gamma^2_{1,h}$ (see Fig. 7.). Denote by s_{ij_x} the point of intersection of the boundary Γ and the ray $(z_{i-2,j},z_{ij}]$, and by s_{ij_y} denote the point of intersection of the boundary Γ and the ray $(z_{i,j+2},z_{ij}]$. At the beginning assume that the solution $u(x,y)$ is determined in the points $z_{i-2,j}$, $z_{i,j+2}$ and write down five-point equation (4.9). Then by means of formula (3.1) eliminate the points belonging to $\gamma^2_{1,h}$. As a result, we obtain seven-point equation with *the second asymmetric X-shaped stencil*:

$$
\begin{aligned}
&\left(\frac{1}{h^2}+\frac{3\delta_x}{(\delta_y+2)4h^2}+\ \frac{3\delta_y}{(\delta_y+2)4h^2}+d(z_{ij})\right)u^h(z_{ij})\\
&-\frac{3\delta_y}{(\delta_y+1)4h^2}u^h(z_{i,j+1})-\frac{3\delta_x}{(\delta_x+1)4h^2}u^h(z_{i-1,j})\\
&-\frac{\delta_y}{(\delta_y+3)4h^2}u^h(z_{i,j-1})-\frac{\delta_x}{(\delta_x+3)4h^2}u^h(z_{i+1,j})\\
&-\frac{1}{4h^2}u^h(z_{i,j-2})-\frac{1}{4h^2}u^h(z_{i,j+2})\\
&=f(z_{ij})+\frac{1}{4h^2}\varphi_{1f}(\delta_x)g(s_{ij_x})+\frac{1}{4h^2}\varphi_{1f}(\delta_y)g(s_{ij_y})
\end{aligned}
\tag{4.11}
$$

where $\delta_x=\rho_1(z_{i-2,j})$, $\delta_y=\rho_2(z_{i,j+2})$.

In the rest of the nodes $z_{ij}\in\gamma^1_{2,h}$ the equations are constructed according to the following principle. Let $z_{ij}\in\gamma^1_{2,h}$, consequently, one adjacent node belongs to $\gamma^1_{1,h}$. Then in the point z_{ij} an equation similar to (4.6) can be constructed with elimination of the point belonging to $\gamma^1_{1,h}$.

Thus, in the result of these constructions a system of linear algebraic equations is obtained, which unites the equalities (4.3) – (4.8), (4.10), (4.11) taken in corresponding nodes. Write down this system in operator form

$$
A^hu^h=f^h \quad \text{on} \quad \omega_h\setminus\gamma^2_{1,h} \tag{4.12}
$$

with sought for grid function $u^h(z_{ij})$ and known right hand side $f^h(z_{ij})$ with the argument $z_{ij}\in\omega_h\setminus\gamma^2_{1,h}$.

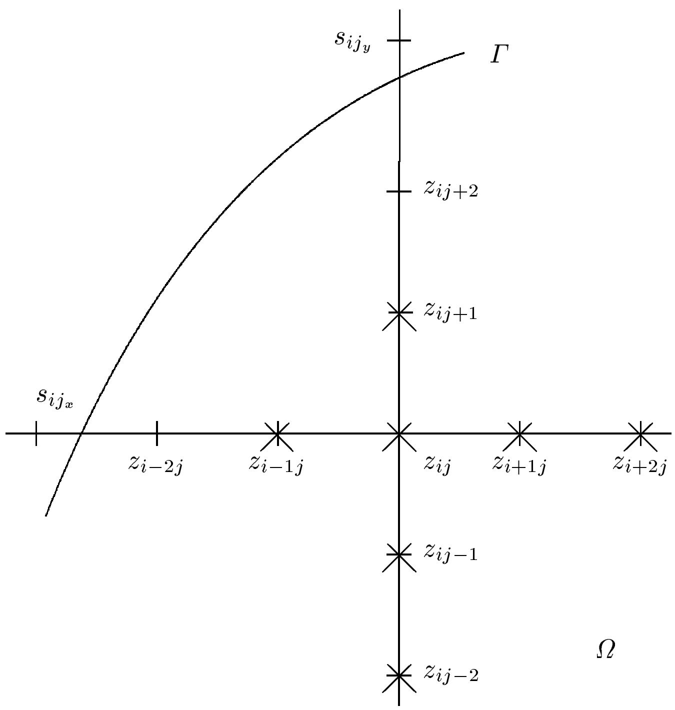


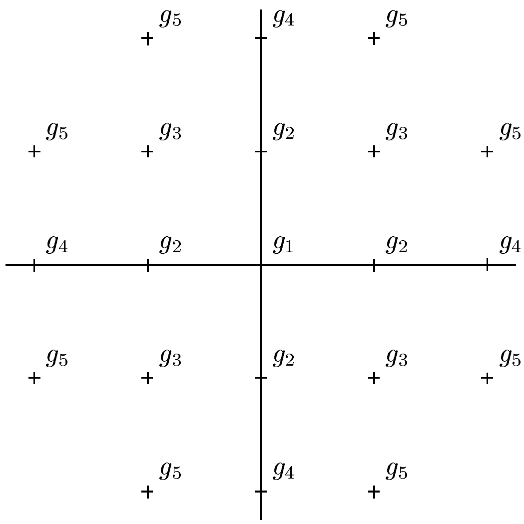
Fig. 7: The second X-shaped asymmetric stencil.

Cross sign marks the nodes of the stencil. $z_{ij} \in \gamma^2_{2,h}$; $z_{i-2,j} \in \gamma^2_{1,h}$; $z_{i,j+2} \in \gamma^2_{1,h}$.

5 Stability, solvability and convergence of the grid problem

Transform the system (4.12) so that its matrix would be M – matrix. At first, enumerate the nodes of the set $\omega_h \setminus \gamma^2_{1,h}$ from 1 to M and give corresponding numbers to the equations in the nodes $z_{ij} \in \omega_h \setminus \gamma^2_{1,h}$ and the variables $u^h(z_{ij})$. In order to utilize the standard results concerning M – matrices, it is necessary that diagonal elements would be positive and off-diagonal ones would be non-negative, and the sum of modules of off-diagonal elements would not exceed a diagonal element. For equations in the nodes $\omega^r_{h,1}$, $\gamma^{31}_{1,h}$,

$\gamma_{1,h}^{33}$, $\gamma_{2,h}$ these conditions are satisfied, but that is not true for equations in the nodes $\gamma_{1,h}^{1}$, $\gamma_{1,h}^{32}$, $\omega_{h,2}^{r}$.



$$g_1 = \frac{2}{h^2}, \quad g_2 = -\frac{1}{10h^2} + \frac{d}{4},$$
$$g_3 = -\frac{3}{10h^2} + \frac{d}{20}, \quad g_4 = 0,$$
$$g_5 = -\frac{1}{20h^2}.$$

Fig. 8: 21-point stencil of the equation in node $z_{ij} \in \omega_{h,2}^{r}$ after the transformation.

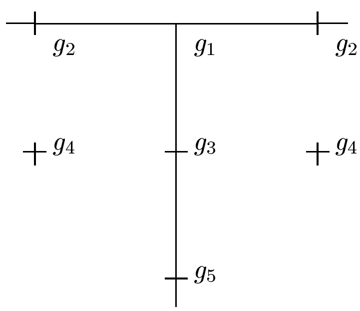
In order to eliminate positive off-diagonal elements, let add to each equation (4.4) in $z_{ij} \in \omega_{h,2}^{r}$ four equations in four regular nodes $z_{i\pm1,j\pm1} \in \omega_{h,1}^{r}$, with weight $a = 1/20$, and four equations in the nodes $z_{i\pm1,j} \in \omega_{h,1}^{r}$, $z_{i,j\pm1} \in \omega_{h,1}^{r}$, with weight $b = 1/4$ (for details see [11]). As a result, in the node $z_{ij} \in \omega_{h,2}^{r}$ we obtain an equation with the stencil shown in Fig. 8 (compare to Fig. 2.b). Then, in addition, require that the following inequality would be true:

$$h^2 \leq 2/\left(5\|d\|_{\infty,\overline{w}_h}\right). \tag{5.1}$$

It is easy to verify that under this condition we come to the following inequalities for coefficients of the new grid equation with extended stencil:

$$
\begin{aligned}
&g_1 \geq 0,\ g_2 \leq 0,\ g_3 \leq 0,\ g_4 = 0,\ g_5 \leq 0,\\
&|g_1(z_{ij})| \geq |g_2(z_{i+1,j}) + g_2(z_{i-1,j}) + g_2(z_{i,j+1}) + g_2(z_{i,j-1})\\
&+g_3(z_{i+1,j+1}) + g_3(z_{i-1,j+1}) + g_3(z_{i+1,j-1}) + g_3(z_{i-1,j-1}) + 8g_5|,
\end{aligned}
$$

which confirm both right signs of coefficients of the stencil and diagonal prevalence.

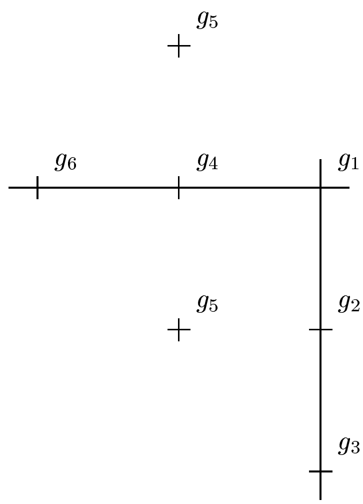


$$
\begin{aligned}
g_1 &= \frac{4}{h^2} + d + \frac{2(-\delta_y^2 + 2\delta_y + 3)}{\delta_y(2+\delta_y)h^2},\\
g_2 &= -\frac{1}{h^2},\\
g_3 &= -\frac{1}{h^2} - \frac{\delta_y^2 - 3\delta_y + 2}{(1+\delta_y)(2+\delta_y)h^2} + \frac{(1-\delta_y)d}{2+\delta_y},\\
g_4 &= -\frac{1-\delta_y}{(2+\delta_y)h^2},\\
g_5 &= 0.
\end{aligned}
$$

Fig. 9: 7-point stencil of the equation in the node $z_{ij} \in \gamma_{1,h}^1$ after the transformation.

Now, let $z_{ij} \in \gamma_{1,h}^1$ and $z_{i,j+1} \in \gamma_{1,h}^{out}$, and $z_{i,j-1} \in \omega_{1,h}^r$. Add to the equation (4.5) in the point z_{ij} one more equation (4.3) in the regular node $z_{i,j-1} \in \omega_{h,1}^r$, with weight $a = (1-\delta_y)/(2+\delta_y)$. As a result, in the point $z_{ij} \in \gamma_{1,h}^1$ we obtain an equation with seven-point stencil, as shown in Fig. 9 (compare to Fig. 3.a). It is easy to verify that under the condition (5.1) and taking into account that $\delta_y \in (0,1]$ we come to the inequalities

$$
\begin{aligned}
&g_1 \geq 0,\ g_2 \leq 0,\ g_3 \leq 0,\ g_4 \leq 0,\ g_5 \leq 0,\\
&|g_1(z_{ij})| > |2g_2 + g_3(z_{i,j-1}) + 2g_4 + g_5|.
\end{aligned}
$$



$$
\begin{aligned}
g_1 &= \frac{4}{h^2} + d + \frac{2(-\delta_x^2 + 2\delta_x + 3)}{\delta_x(2+\delta_x)h^2} - \frac{3\delta_y}{(\delta_y+1)h^2},\\
g_2 &= -\left(\frac{1}{h^2} - \frac{3\delta_y}{(2+\delta_y)h^2}\right),\\
g_3 &= -\frac{\delta_y}{(3+\delta_y)h^2},\\
g_4 &= -\frac{1}{h^2} - \frac{\delta_x^2 - 3\delta_x + 2}{(1+\delta_x)(2+\delta_x)h^2} + \frac{(1-\delta_x)d}{2+\delta_x},\\
g_5 &= -\frac{1-\delta_x}{(2+\delta_x)h^2}, \quad g_6 = 0.
\end{aligned}
$$

Fig. 10: 7-point stencil of the equation in the node $z_{ij} \in \gamma_{1,h}^{32}$ after the transformation.

Let $z_{ij} \in \gamma_{1,h}^{32}$, $z_{i+1,j} \in \gamma_{1,h}^{out}$, and $z_{i-1,j} \in \omega_{h,1}^{r}$. Add to the equation (4.8) in the point z_{ij} one more equation (4.3) in the regular node $z_{i-1,j} \in \omega_{h,1}^{r}$ with weight $a = (1-\delta_x)/(2+\delta_x)$. As a result, in the point $z_{ij} \in \gamma_{1,h}^{32}$ we obtain an equation with seven-point stencil shown in Fig. 10 (compare to Fig. 5). Under the condition (5.1) and taking into account that $\delta_x \in (0,1]$ and $\delta_y \in (0,1]$ we obtain the inequalities

$$
\begin{aligned}
&g_1 \ge 0,\ g_2 \le 0,\ g_3 \le 0,\ g_4 \le 0,\ g_5 \le 0,\ g_6 \le 0,\\
&|g_1(z_{ij})| > |g_2 + g_3 + g_4(z_{i-1,j}) + 2g_5 + g_6|.
\end{aligned}
$$

Thus, we obtain a system consisting of the equations corresponding to the points belonging to $\omega^r_{h,1}$, $\gamma^{31}_{1,h}$, $\gamma^{33}_{1,h}$ and $\gamma_{2,h}$, and transformed equations corresponding to the points belonging to $\omega^r_{h,2}$, $\gamma^1_{1,h}$ and $\gamma^{32}_{1,h}$. With the regard for the signs of diagonal and off-diagonal elements, diagonal prevalence and indecomposability [9], the matrix of the transformed system is M–matrix. The obtained equivalent grid problem can be written down as

$$B^h u^h = g^h \quad \text{on} \quad \omega_h \setminus \gamma^2_{1,h} \tag{5.2}$$

with the same unknown grid function u^h but with transformed right-hand side g^h.

Theorem 36. *Let the condition* (1.3) *be satisfied and the step h be small enough:*

$$h^2 \le 2/(5\|d\|_{\infty,\overline{\Omega}}). \tag{5.3}$$

Then for arbitrary right-hand side f^h the solution of the problem (4.12) *satisfies the estimate*

$$\|u^h\|_{\infty,\overline{\omega}_h\setminus\gamma^2_{1,h}} \le \frac{11}{48}\|f^h\|_{\infty,\omega^r_h} + \|f^h/S^h\|_{\infty,\omega^{ir}_h}, \tag{5.4}$$

where $S^h(z_{ij})$ is the sum of coefficients of the grid equation (4.12) *in the node z_{ij} and is strictly positive on ω^{ir}_h.*

Proof. Introduce a function

$$w_1(x,y) = c_1 x(1-x), \quad c_1 = \frac{11}{12}\|f^h\|_{\infty,\omega^r_h}. \tag{5.5}$$

Note that derivatives of the order 3 and higher of this function are equal to zero. Therefore the exact approximation of the difference operators L^h, L^{2h} and interpolation formulas is attained for this function. From this, under the condition (1.3) we have

$$L^h w_1 = L w_1 = d w_1 + 2c_1 \ge 2c_1 > 0 \quad \text{on} \quad \omega_h, \tag{5.6}$$

$$L^{2h} w_1 = L w_1 = d w_1 + 2c_1 \ge 2c_1 > 0 \quad \text{on} \quad \omega_{00}. \tag{5.7}$$

Taking into account (5.6), in regular nodes of the first kind we obtain

$$B^h w_1 = L w_1 \ge \frac{11}{6}\|f\|_{\infty,\omega^r_h} \ge A^h u^h = B^h u^h \quad \text{on} \quad \omega^r_{h,1}. \tag{5.8}$$

Similar expression for regular nodes of the second kind can be obtained by means of the rule of transformation of the operator A^h into B^h (detailed computations see in [11]):

$$B^h w_1 \geq \frac{12}{5} c_2 \geq \frac{11}{5} \|f\|_{\infty,\omega_h^r} \geq B^h u^h \quad \text{on} \quad \omega_{h,2}^r. \tag{5.9}$$

In irregular nodes, from the analysis of the rules of transformation of A^h into B^h (i.e., possible addition of a regular equation with weight $\leq 1/2$) with account of (5.6) or (5.7) it follows that

$$B^h w_1 \geq L w_1 \geq 2c_1 \quad \text{on} \quad \omega_h^{ir}. \tag{5.10}$$

Introduce a constant function

$$w_2(x,y) = c_2 \quad \text{where} \quad c_2 = \|f^h / S^h\|_{\infty,\omega_h^{ir}}. \tag{5.11}$$

After the substitution of it into the operator B^h, two possible situations take place: either coincidence or lack of coincidence of the equations for z_{ij} in (5.2) and (4.12). In the first case (when $z_{ij} \in \omega_{h,1}^r \cup \gamma_{1,h}^{31} \cup \gamma_{1,h}^{33} \cup \gamma_{2,h}$) we obtain

$$B^h w_2(z_{ij}) = A^h w_2(z_{ij}) = c_2 S^h(z_{ij}) \geq 0, \tag{5.12}$$

in irregular nodes being diagonal prevalence and the value of $S^h(z_{ij})$ being strictly positive. In the second case the equation in (5.2) is obtained from (4.12) by addition of regular equations with positive weights. Therefore (for $z_{ij} \in \omega_{h,2}^r \cup \gamma_{1,h}^1 \cup \gamma_{1,h}^{32}$) we have

$$B^h w_2(z_{ij}) \geq A^h w_2(z_{ij}) = c_2 S^h(z_{ij}) \geq 0, \tag{5.13}$$

the value of diagonal prevalence being not less in irregular nodes and $S^h(z_{ij})$ being strictly positive again. So, combining the inequalities (5.12) and (5.13) we obtain

$$B^h w_2(z_{ij}) \geq 0 \quad \text{on} \quad \omega_h^r, \tag{5.14}$$

$$B^h w_2(z_{ij}) \geq c_2 S^h(z_{ij}) \geq f^h(z_{ij}) \quad \text{on} \quad \omega_h^{ir}. \tag{5.15}$$

Summing up the inequalities (5.8) and (5.9) with (5.19), we come to the following expression in regular nodes

$$B^h(w_1 + w_2) \geq B^h u^h \quad \text{on} \quad \omega_h^r. \tag{5.16}$$

In irregular nodes this expression is obtained by summation of (5.10) with (5.15) and taking into account the rule of transformation of A^h into B^h:

$$B^h(w_1 + w_2) \geq 2c_1 + c_2 \geq 2c_1 + A^h u^h \geq B^h u^h \quad \text{on} \quad \omega_h^{ir}.$$

From two last inequalities on the basis of the comparison theorem [9] it follows that

$$w_1 + w_2 \geq u^h \quad \text{on} \quad \omega_h \setminus \gamma^2_{1,h}.$$

After the replacement of f^h with $-f^h$ the above reasonings give the evaluation

$$w_1 + w_2 \geq -u^h \quad \text{on} \quad \omega_h \setminus \gamma^2_{1,h}.$$

The two last evaluations can be combined into the inequality

$$|u^h| \leq w_1 + w_2 \quad \text{on} \quad \omega_h \setminus \gamma^2_{1,h}.$$

After taking maximum in the right-hand side over $[0,1] \times [0,1]$ we come to the evaluation (5.4). □

Theorem 1 conveys stability of the problem (4.12) with respect to the boundary values and right-hand side, and, besides that, from it naturally follows unique solvability, since corresponding to it uniform system admits only zero solution.

Lemma 1. *If the conditions* (1.3) *and* (5.3) *are satisfied, then there exists a constant c_3 independent from h and domain Ω, such that the value $c_3 S^h(z_{ij})$ in irregular node $z_{ij} \in \omega_h^{ir}$ majorizes the modules of all non-zero coefficients of the grid equation* (4.12) *corresponding to this node.*

Proof. Let consider in details only one variant, for instance, the equation (4.5) in the node $z_{ij} \in \gamma^1_{1,h}$. Computation of $S^h(z_{ij})$ with the account of (1.3) and (3.2) gives the evaluation

$$S^h(z_{ij}) = d(z_{ij}) + \frac{1}{h^2}\varphi_{2f}(\delta_y) \geq \frac{6}{\delta_y(1+\delta_y)(2+\delta_y)h^2}. \tag{5.17}$$

For any $\delta_y \in (0,1]$ we have

$$S^h(z_{ij}) \geq \frac{1}{\delta_y h^2} \geq \frac{1}{h^2}. \tag{5.18}$$

From this and (5.3) we obtain

$$\frac{2}{5} S^h(z_{ij}) \geq \frac{2}{5h^2} \geq d(z_{ij}). \tag{5.19}$$

Except that, from (5.18) follow the inequalities

$$3S^h(z_{ij}) \geq \frac{3}{\delta_y h^2} \geq \frac{3(1-\delta_y)}{\delta_y h^2}.$$

By summation of the three last inequalities (the first with the factor 4), we obtain

$$\frac{37}{5}S^h(z_{ij}) \geq \frac{4}{h^2} + d(z_{ij}) + \frac{3(1-\delta_y)}{\delta_y h^2}. \tag{5.20}$$

Thus, the expression in the left-hand side majorizes the positive diagonal coefficient. It is easy to verify that it majorizes the modules of the other four coefficients of the equation (4.5).

So, the statement of the lemma is proved for the nodes $\gamma^1_{1,h}$ with constant 37/5. Similarly to the reasonings in (5.17) – (5.20), the existence of such constants for other kinds of irregular nodes can be proved. Denoting the maximal of them by c_3, we complete the proof of the Lemma. □

Corollary 1. Looking through the equations (4.5) – (4.8), (4.10) and (4.11) one can make sure that each of them contains a coefficient with absolute value not less than $1/h^2$ (in (4.7), (4.8), (4.10) and (4.11) that is diagonal coefficient). Therefore from Lemma 1 it follows that

$$S^h(z_{ij}) \geq \frac{1}{c_3 h^2} \quad \text{for} \quad z_{ij} \in \omega_h^{ir}. \tag{5.21}$$

Theorem 37. *Let u, u^h be the solutions of the problems* (1.1) – (1.2) *and* (4.12)*, respectively, and the conditions* (1.3), (1.4), (5.3) *be satisfied. Then*

$$\|u - u^h\|_{\infty,\overline{\omega}_h \setminus \gamma^2_{1,h}} \leq Ch^4 \tag{5.22}$$

where constant C is independent of h.

Proof. Let show that the solution u^h can be represented in the form:

$$u^h = u + h^4\rho^h \quad \text{on} \quad \omega_{11} \setminus \gamma^2_{1,h}, \tag{5.23}$$

$$u^h = u + h^4 w_{01} + h^4\rho^h \quad \text{on} \quad (\omega_{01} \cup \omega_{10}) \setminus \gamma^2_{1,h}, \tag{5.24}$$

$$u^h = u + h^4 w_{00} + h^4\rho^h \quad \text{on} \quad \omega_{00} \setminus \gamma^2_{1,h}, \tag{5.25}$$

(5.26)

where the functions

$$w_{01} = -\frac{1}{48}\mu, \quad w_{00} = -\frac{1}{12}\mu, \quad \mu = \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \tag{5.27}$$

do not depend on h, and the remainder term ρ^h is limited in the following way:

$$\|\rho^h\|_{\infty,\overline{\omega}_h \setminus \gamma^2_{1,h}} \leq c_4. \tag{5.28}$$

The proof of the representations (5.23) – (5.27) is obtained by complication of proof of the Theorem 4 from the work [11]. Indeed, on the basis of the computations given there one can obtain equalities in regular nodes ω_h^r

$$A^h\rho^h = L^h\rho^h = \xi^h \quad \text{on} \quad \omega_{h,1}^r, \tag{5.29}$$

$$A^h\rho^h = L^h\rho^h - L^{2h}\rho^h = \xi^h \quad \text{on} \quad \omega_{h,2}^r \tag{5.30}$$

with a grid function

$$|\xi^h| \le c_5 \quad \text{on} \quad \omega_h^r. \tag{5.31}$$

Let consider in details the situation in irregular nodes after the example of the grid equation (4.5) in the node $z_{ij} \in \gamma_{1,h}^1$. Substitute the expansions (5.23) – (5.25) into the expression $L^h u^h(z_{ij})$ and for the function u perform the expansion into Taylor series with respect to z_{ij} with the remainder term of the order h^4. In the node $z_{i,j+1}$ lying outside $\overline{\Omega}$ the value of $u^h(z_{i,j+1})$ is determined as (one-dimensional) Taylor series with respect to s_{ij} up to the derivative $\partial^4 u/\partial y^4$ inclusive. Then for the function u interpolation formula (3.2) with remainder term (3.3) and multiplied by $1/h^2$ is used. As a result, we obtain the equality

$$A^h u(z_{ij}) = f^h(z_{ij}) + h^2\zeta(z_{ij}) \tag{5.32}$$

with the evaluation of the remainder term

$$|\zeta(z_{ij})| \le c_6. \tag{5.33}$$

Consider terms of the form $h^4 w_{01}$ and $h^4 w_{00}$ in the expansions (5.24), (5.25). On the basis of (5.27) they are evaluated as

$$h^4 \max_{\overline{\Omega}} \{|w_{00}|, |w_{01}|\} \le \frac{h^4}{12}\|\mu\|_{\infty,\overline{\Omega}}. \tag{5.34}$$

On the basis of Lemma 1, under any possible arrangement of these terms on the stencil of the equation (4.5) (see Fig. 3a) the result $\eta(z_{ij}))$ of linear combination of these terms with corresponding coefficients of the grid equation (4.5) can be evaluated as

$$|\eta(z_{ij})| \le 5c_3 S^h(z_{ij})\frac{h^4}{12}\|\mu\|_{\infty,\overline{\Omega}}. \tag{5.35}$$

Thus, after substitution of (5.23) – (5.25) into the expressions $A^h u^h(z_{ij})$ reduce a part of terms due to (4.12) and (5.32), divide the others terms by h^4 and group together the terms with $\zeta(z_{ij})$ and $\eta(z_{ij})$ into one remainder term ξ^h:

$$A^h\rho^h(z_{ij}) = \xi^h(z_{ij}), \quad z_{ij} \in \gamma_{1,h}^1. \tag{5.36}$$

Due to (5.33), (5.35), and Corollary 1 we have the evaluation

$$|\xi^h(z_{ij})| \le h^{-2}|\zeta(z_{ij})| + h^{-4}|\eta(z_{ij})| \le c_3(c_6 + 5/12\ \|\mu\|_{\infty,\overline{\Omega}})S^h(z_{ij}). \quad (5.37)$$

Similar expressions are obtained in other kinds of irregular nodes. Finally,

$$A^h\rho^h = \xi^h \quad \text{on} \quad \omega_h^{ir} \qquad (5.38)$$

with a grid function ξ^h for which the following evaluation is valid:

$$|\xi^h(z_{ij})| \le c_7 S^h(z_{ij}), \quad z_{ij} \in \omega_h^{ir}, \qquad (5.39)$$

where $c_7 = c_3(c_6 + 7/12\ \|\mu\|_{\infty.\overline{\Omega}})$.

In the end we arrive at the system of equations (5.29), (5.30), and (5.38), which uniquely determines the grid function ρ^h. On the basis of Theorem 1 we obtain the evaluation

$$\|\rho^h\|_{\infty,\overline{\omega}_h\setminus\gamma_{1,h}^2} \le \frac{11}{48}\|\xi^h\|_{\infty,\omega_h^r} + \|\xi^h/S^h\|_{\infty,\omega_h^{ir}}. \qquad (5.40)$$

From it, due to (5.31) and (5.39), (5.28) follows with constant

$$c_4 = 11/48\ c_5 + c_7.$$

The final affirmation of (5.22) follows from (5.23) – (5.25) with use of (5.28) and (5.34). □

6 Numerical examples

As in [11], let apply the constructed method to two problems of the form (1.1) – (1.2) with improved smoothness and with oscillating solution. Let the domain Ω be bounded by a circumference Γ with center in point $(0.5, 0.5)$ and radius 0.49.

The problem I has the form

$$
\begin{aligned}
-\Delta u = {} & 2\cos\left(\frac{\pi x}{2}\right) y(1-y)\cos\left(\frac{\pi y}{2}\right) \\
& + (1-x)\sin\left(\frac{\pi x}{2}\right)\pi y(1-y)\cos\left(\frac{\pi y}{2}\right) \\
& - x\sin\left(\frac{\pi x}{2}\right)\pi y(1-y)\cos\left(\frac{\pi y}{2}\right) \\
& + \frac{1}{2}x(1-x)\cos\left(\frac{\pi x}{2}\right)\pi^2 y(1-y)\cos\left(\frac{\pi y}{2}\right) \\
& + 2x(1-x)\cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right) \qquad \text{(I)} \\
& + x(1-x)\cos\left(\frac{\pi x}{2}\right)(1-y)\sin\left(\frac{\pi y}{2}\right)\pi \\
& - x(1-x)\cos\left(\frac{\pi x}{2}\right) y\sin\left(\frac{\pi y}{2}\right)\pi \quad \text{in} \quad \Omega, \\
u = {} & g \quad \text{on} \quad \Gamma
\end{aligned}
$$

with a function g being equal on Γ to the exact solution

$$
u(x,y) = x(1-x)\cos\left(\frac{\pi x}{2}\right) y(1-y)\cos\left(\frac{\pi y}{2}\right).
$$

The problem II has the form

$$
\begin{aligned}
-\Delta u = {} & -32c_x(1-x)y(1-y) + 512s_x x(1-x)y(1-y) \\
& +32c_x xy(1-y) + 2s_x y(1-y) - 32c_x x(1-x)(1-y) \qquad \text{(II)} \\
& +32c_x x(1-x)y + 2s_x x(1-x) \quad \text{in} \quad \Omega, \\
u = {} & g \quad \text{on} \quad \Gamma
\end{aligned}
$$

where $c_x = \cos(16x+16y)$ and $s_x = \sin(16x+16y)$. The function g on Γ is equal to the exact solution as well $u(x,y) = \sin(16x+16y)x(1-x)y(1-y)$.

In Fig. 11 a quarter of the domain Ω is shown for $N = 44$.

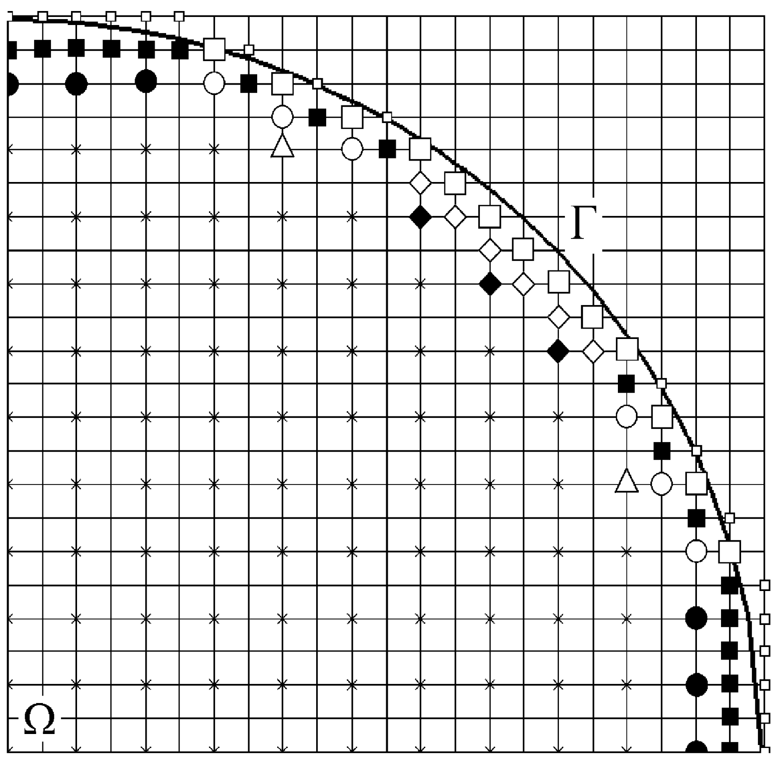


Fig. 11: Scheme of possible arrangement of kinds of nodes on the grid ω_h;

Here new symbols are introduced:

+ — $z_{ij} \in \omega^r_{h,1}$; ✱ — $z_{ij} \in \omega^r_{h,2}$; ⌑ — $z_{ij} \in \gamma^{out}_{1,h}$;
■ — $z_{ij} \in \gamma^1_{1,h}$; □ — $z_{ij} \in \gamma^2_{1,h}$; ● — $z_{ij} \in \gamma^1_{2,h}$;
○ — $z_{ij} \in \gamma^{32}_{1,h} \cup \gamma^{33}_{1,h}$; △ — $z_{ij} \in \gamma^3_{2,h}$; ◇ — $z_{ij} \in \gamma^{31}_{1,h}$;
◆ — $z_{ij} \in \gamma^2_{2,h}$.

The data of the numerical experiment are presented in Table 1.

Table 1: Error of approximate solutions of the problem with improved smoothness.

N	method of the fourth order		method of the second order	
	Ψ_1	Ψ_2	Ψ_1	Ψ_2
10	$5.84_{10}-04$	$1.68_{10}-04$	$4.39_{10}-04$	$2.04_{10}-04$
14	$6.32_{10}-05$	$1.50_{10}-05$	$1.84_{10}-04$	$8.40_{10}-05$
18	$4.24_{10}-05$	$9.22_{10}-06$	$1.07_{10}-04$	$4.79_{10}-05$
20	$1.72_{10}-05$	$6.09_{10}-06$	$8.64_{10}-05$	$3.86_{10}-05$
28	$4.32_{10}-06$	$1.31_{10}-06$	$4.37_{10}-05$	$1.95_{10}-05$
30	$2.18_{10}-06$	$4.79_{10}-07$	$3.80_{10}-05$	$1.70_{10}-05$
32	$4.31_{10}-06$	$9.11_{10}-07$	$3.35_{10}-05$	$1.49_{10}-05$
36	$1.55_{10}-06$	$3.85_{10}-07$	$2.64_{10}-05$	$1.18_{10}-05$
40	$1.02_{10}-06$	$2.77_{10}-07$	$2.13_{10}-05$	$9.48_{10}-06$
56	$2.52_{10}-07$	$5.29_{10}-08$	$1.08_{10}-05$	$4.82_{10}-06$
60	$2.88_{10}-07$	$4.87_{10}-08$	$9.17_{10}-06$	$4.07_{10}-06$
64	$2.59_{10}-07$	$4.29_{10}-08$	$7.09_{10}-06$	$3.10_{10}-06$

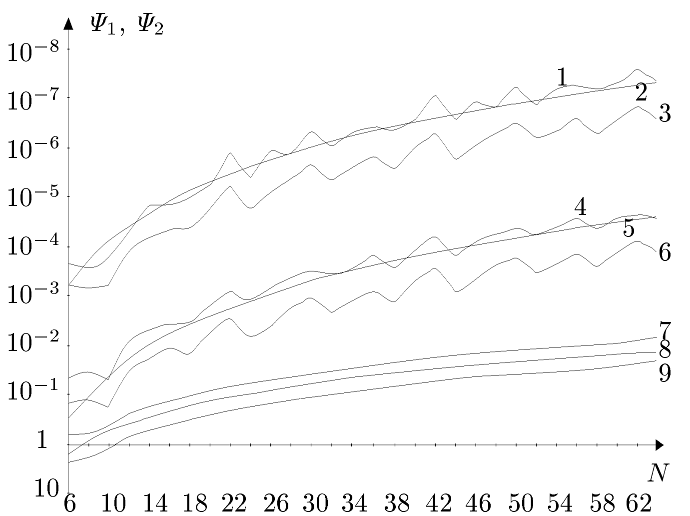


Fig. 12: Error of approximate solutions of the problems I and II.

In Fig. 12 the results of numerical experiments are shown in logarithmic scale over the $Y-$axis. The numbers 1, 4 and 7 mark mean square error

$$\Psi_1 = \|u - u^h\|_{2,\overline{\omega}_h \setminus \gamma^2_{1,h}} = \left(\sum_{z \in \overline{\omega}_h \setminus \gamma^2_{1,h}} \left(u(z) - u^h(z)\right)^2 \right)^{1/2}$$

for the problems I and II, solved by the proposed in the present paper method, and for the problem I, solved by a standard method with the second order of accuracy, respectively [2], [6]. The numbers 3, 6 and 9 mark uniform errors

$$\Psi_2 = \|u - u^h\|_{\infty,\overline{\omega}_h\setminus\gamma^2_{1,h}}$$

for the problems I and II, solved by the method proposed in the present paper, and for the problem I, solved by a standard method with the second order of accuracy, respectively. The numbers 2, 5 and 8 mark diagrams of the curves $\delta = c_1h^4$, $\delta = c_2h^4$ and $\delta = h^2$, respectively.

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